. Cameron Math 61 Friday, March 22nd, 2019

Final

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Section:

	Tuesday:	Thursday:	
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Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators**, books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

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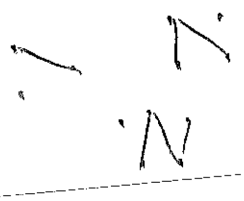
Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$ $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k}$
      (a) $\frac{(n+k)!}{k!n!}$
      (b) $\frac{(n+1)!}{k!(n+1-k)!}$
      (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$
      (d) none of the above

   2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:
      n! term vertices $2^h \geq n!$
      $h \geq \lg(n!)$
      (a) a height of $\geq \lg(n!)$
      (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)
      (c) a height of $O(\lg(n!))$
      (d) a height of $O(n \lg n)$

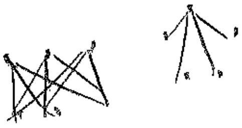
   3. If G is a graph with n vertices and $n-2$ edges, then:
      (a) G is a tree
      (b) G is connected
      (c) G is disconnected
      (d) G is simple



Question 1 continues on the next page...

Question 1 continued...

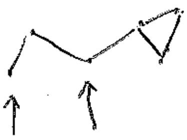
4. Which of these graphs has an Euler cycle?
   (a) K_4
   (b) K_5
   (c) $K_{3,3}$
   (d) $K_{2,3}$



5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.
   (a) Less than or equal to n
   (b) More than n but less than or equal to $n^2/2$
   (c) More than $n^2/2$ but less than or equal to n^2
   (d) More than n^2

$\leq n-1$

initialize 0 and ∞
circle lowest
update all connected



$S_n - S_{n-1} - 6S_{n-2} = 0$ $\quad (r-3)(r+2)=0$

$x^n - x^{n-1} - 6x^{n-2} = 0 \quad x^{n-2}(x^2 - x - 6) = 0$

$S_n = A\cdot 3^n + B\cdot 2^n$

$S_0 = A\cdot 3^0 + B\cdot 2^0 = 2$

$A+B=2$

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

$S_1 = A\cdot 3 + B\cdot 2 = 1$

$3A+2B=1$

2, 1, 13, 18

$3A+2B=1$

$-\ 2A+2B=4$

$A=-3$

$B=5$

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}?

$$5\cdot 2^{100} - 3\cdot 3^{100}$$

$S_n = -3\cdot 3^n + 5\cdot 2^n$

$S_n = 5\cdot 2^n - 3^{n+1}$

~~$5\cdot 2^1 - 3\cdot 3^1$~~

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

^M^M^M^M^M^M^M^ $\quad 7!\cdot P(8,4)\ /7$

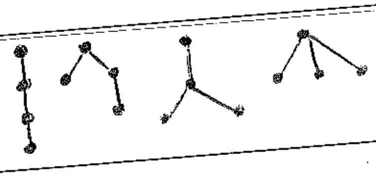
$$6!\cdot P(8,4)$$

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

20 stars, 4 bars $\quad$ 1|2|3|4|5

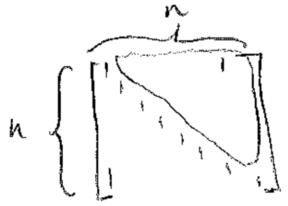
$$\binom{24}{4}$$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.



(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

$$2^{\frac{n(n-1)}{2}}$$



$2^{(n-1)+(n-2)+\cdots+1}$ $\quad 2^{\frac{n(n-1)}{2}}$ $\quad 4+3+2+1$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$. integer

(a) (4 points) Show that C is an equivalence relation. Sym., refl., trans.

1. Reflexive: For all $x \in \mathbb{R}$, $x - x = 0$, which is an integer ($0 \in \mathbb{Z}$).
   So $(x, x) \in C$. ✓

2. Symmetric: For all $x, y \in \mathbb{R}$, if $(x, y) \in C$, then $x - y \in \mathbb{Z}$.
   Since $(x-y)$ is an integer, $-(x-y)$ is also an integer, which is $y - x$.
   So $(y, x) \in C$. Therefore if $(x, y) \in C$, then $(y, x) \in C$. ✓

3. Transitive: For all $x, y, z \in \mathbb{R}$, if $(x, y) \in C$, then $x - y \in \mathbb{Z}$, and if $(y, z) \in C$, then $y - z \in \mathbb{Z}$.
   $(x - y) + (y - z) = x - \cancel{y} + \cancel{y} - z = x - z$
   Since adding two integers results in an integer, $x - z \in \mathbb{Z}$.
   So $(x, z) \in C$. Therefore if $(x, y) \in C$ and $(y, z) \in C$, then $(x, z) \in C$. ✓

Since C is reflexive, symmetric, and transitive, C is an equivalence relation.

Question 3 continues on the next page...

$-1, \dots$

$[1] : \{1, 2, 3, 4 \dots\}$

eq class of C : $[1.5] : \{-3.5, -2.5 \dots \; 1.5, 2.5, \dots\}$

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$. Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

f = "add 0.5"

$\tilde{f}$ brings to a diff eq class

$\tilde{f} : \tilde{\mathbb{R}} \to \tilde{\mathbb{R}}$

$x = y$, $\tilde{f}(x) = \tilde{f}(y)$ and $\tilde{f}(x)$ always defined

1. Defines everywhere : For every $[a] \in \tilde{\mathbb{R}}$, $f(a) \in \mathbb{R}$ so $[f(a)]$ exists.

2. Unique : Let $[x], [y] \in \tilde{\mathbb{R}}$, such that $x = y$ (so $[x] = [y]$)

Suppose that $\tilde{f}([x]) \neq \tilde{f}([y])$.

Then $[f(x)] \neq [f(y)]$. Since f is a function, $f(x) = f(y)$.

But the equivalence classes of two equal values must be the same, so this is a contradiction, and the statement is wrong, and $\tilde{f}([x]) = \tilde{f}([y])$ for all $[x] = [y]$.

Since $\tilde{f}$ is defined everywhere and gives a unique answer for each value, it's a function.

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

Given 2 eq. classes, pick one from $[a]$ and it must be = to any from $[b]$

$g(x) = x + 1$

~~Then, $([x], [0]) \in \tilde{g}$ for all $[x] \in \tilde{\mathbb{R}}$. So~~

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?

$\boxed{(m-1)i+1}$

$t=9$
$i=4$

$3i$

i internal vertices → i parent vertices → mi children → $mi - i$ children excluding the internal vertices → $mi - i + 1$ adding the only point that is not a child (root).

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \leq m^h$.

I want to generate every path from the root to a terminal vertex. Since T is a tree, these paths are unique. At each internal vertex I have m choices of which branch to follow. I will visit up to h vertices on each path because h is the length of the largest path in T. So there are up to m^h different paths.

Since there is a bijection from paths to terminal vertices, the number of paths is the number of terminal vertices. So there are up to m^h terminal vertices. So $t \leq m^h$. ✓

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

~~Since every vertex in G is in its spanning tree, starting a minimal spanning tree algorithm at any vertex.~~

Pick a vertex v in G and start Prim's algorithm at that vertex. Since it always adds the edge with minimal weight from each vertex, and all the weights are distinct, no arbitrary choices have to be made, so there is only one minimal spanning tree that includes v.

Since every spanning tree includes v, there is only one minimal spanning tree of G.

Question 5 continues on the next page...

(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.

Let A, B be two distinct spanning trees of G. Suppose that they have the same total weight.

All spanning trees of G must have the same number of edges,

So the weight sum of A and B have the same number of terms.

$\boxed{\text{No such graph exists}}$

6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

Each face must be bounded by 3 or more edges because a face is a cycle, so the number of boundary segments is at least $3F$.

Each edge is a part of up to 2 faces, so the number of boundary segments is at most $2E$.

So $2E \geq$ (boundary segments) $\geq 3F$, so $2E \geq 3F$.

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

$2E \geq 3F$ and $V - E + F = 2$.

$F \leq \frac{2}{3}E$

$V - E + \frac{2}{3}E \geq 2$

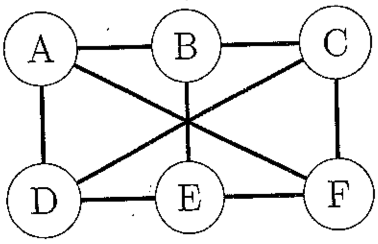
$V - \frac{1}{3}E \geq 2$

$3V - E \geq 6 \rightarrow 3V - 6 \geq E$

$E \leq 3V - 6$

Question 6 continues on the next page...

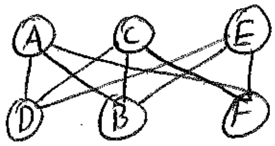
(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.



A, C, E

D, B, F

This graph can be redrawn as



which makes it clear that it's a complete bipartite graph on 3 and 3 vertices ($K_{3,3}$) with the parts being $\{A, C, E\}$ and $\{D, B, F\}$.

Since $K_{3,3}$ is not planar, this graph is $\boxed{\text{NOT PLANAR.}}$

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

<u>Base case</u>: $n = 1$. $7^1 - 1 = 6$, which is divisible by 6. ✓

<u>Inductive step</u>: Assume $7^n - 1$ is divisible by 6.
Try to show $7^{n+1} - 1$ is divisible by 6.

$$7^{n+1} - 1 = 7 \cdot 7^n - 1 = (6+1)\,7^n - 1 = 6 \cdot 7^n + 7^n - 1$$

This is the sum of $6 \cdot 7^n$ and $7^n - 1$.

- $6 \cdot 7^n$ is divisible by 6 because it is 6 times an integer.
- $7^n - 1$ is divisible by 6 by the inductive hypothesis.
- A sum of two numbers divisible by 6 is also divisible by 6. So $6 \cdot 7^n + 7^n - 1 = 7^{n+1} - 1$ is divisible by 6. ✓

$7(7^n) - 1$
$(6+1)\,7^n - 1$
$6 \cdot 7^n + 7^n - 1$

$\cancel{7^n \cdot 7 - 6}$
$\cancel{7^n(6+1) - 6}$
$\cancel{6 \cdot 7^n + 7^n - 6}$
$\cancel{6(7^n - 1) + 7^n}$

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7. pigeonhole principle

1 mod 7 = 1
11 mod 7 = 4
111 mod 7 = ~~?~~
1111 mod 7

Pigeonholes: $\{1, 11, 111, \ldots\}$
Pigeons: numbers divisible by 7.

For any d-digit number n, there are d pigeonholes ~~[illegible]~~ less than or equal to n, and $\lfloor \frac{n}{d} \rfloor$ pigeons less than or equal to n. ~~[illegible]~~

Since n is arbitrarily large, $\lfloor \frac{n}{d} \rfloor > d$, so there are more pigeons than pigeonholes. So there's a number consisting of all 1's that's divisible by 7.

$\frac{n}{d} > d$
$n > d^2$

1, 2, 4, 7

0, 1, 1, 2, 3, 5, 8

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

h=2

$V_0 = 1$
$V_1 = 2$
$V_2 = 4$

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$

A balanced binary tree of height n that is minimal will be composed of its root vertex, the minimal number of vertices in its left subtree (which is of height one less), and the minimal number of vertices in its right subtree (which is of height one less than the left subtree; this keeps the tree balanced in the minimal way). Left and right are interchangeable.

So $V_n = V_{n-1} + V_{n-2} + 1$.

- root : 1
- left (or right) : V_{n-1}
- right (or left) : V_{n-2}

total : V_n

(b) (3 points) Show that for $n \geq 0$, $v_n = F_{n+3} - 1$ where F_k is the k^{th} Fibonacci number.

Since $V_0 = 1 = F_3 - 1$ and $V_1 = 2 = F_4 - 1$, the initial conditions of V_n are the initial conditions of $F_{n+3} - 1$.

$V_n = V_{n-1} + V_{n-2} + 1$

$V_n - 1 = V_{n-1} + V_{n-2}$ → $V_{n+3} - 1 = V_{n+2} + V_{n+1}$ → this is the recurrence relation for $F_{n+3} - 1$.

So V_n is the Fibonacci sequence, minus 1, and shifted in its initial conditions by 3. ✓

Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

Since $V_n = F_{n+3} - 1$, $V_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+3} - 1$

For large n, $V_n \le \frac{1}{\sqrt{5}}\phi^{n+3}$, so V_n is $O(\phi^{n+3})$.

For large n, $V_n \ge 0.00001 \cdot \phi^{n+3}$, so V_n is $\Omega(\phi^{n+3})$.

Since V_n is $O(\phi^{n+3})$ and $\Omega(\phi^{n+3})$, V_n is $\Theta(\phi^{n+3})$.

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

$3^n = (2+1)^n$ ~~$= \sum_{i=0}^{n} \binom{n}{i} 2^i 1^n = \sum_{i=0}^{n} \binom{n}{i} 2^i$~~ ~~Binomial Theorem~~

$(2+1)^n = \sum_{i=0}^{n} \binom{n}{i} 2^i \cdot 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i} 2^i$ ✓

Binomial Theorem

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

~~$\frac{n!}{i!(n-i)!}\frac{m!}{(r-i)!(m-r+i)!}$~~ $\binom{n+m}{r}$ is the number of ways to choose r elements from $n+m$ elements.

You could do this by choosing all the elements from the last m elements, or one element from the first n and the rest from the last m, or .... all the elements from the first n elements.

By the addition principle, since all of the above result in non-overlapping selections, you can add them up to get $\binom{n+m}{r}$.

So $\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots \binom{n}{r}\binom{m}{0}$

$= \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$ ✓

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