Math 61-1 Final exam

1.1 2 / 2

✓ - **0 pts** **Correct (c)**

- **2 pts** Incorrect

1.2 0 / 2

- **0 pts** Correct (a)

✓ - **2 pts** **Incorrect**

1.3 2 / 2

✓ - **0 pts** **Correct (c)**

- **2 pts** Incorrect

1.4 2 / 2

✓ - **0 pts** **Correct (b)**

- **2 pts** incorrect

1.5 2 / 2

✓ - **0 pts** **Correct (a)**

- **2 pts** Incorrect

QUESTION 2

Short answer 10 pts

2.1 2 / 2

✓ - **0 pts** **Correct ((-2)^100 + 3^100)**

- **1 pts** Almost correct (small arithmetic error in answer)

- **2 pts** Incorrect

2.2 0 / 2

- **0 pts** Correct (C(7,4)6!4!)

- **1 pts** Close

✓ - **2 pts** **Incorrect**

2.3 2 / 2

✓ - **0 pts** **Correct (24C4)**

- **1 pts** Close

- **2 pts** incorrect

2.4 2 / 2

✓ - **0 pts** **Correct**

- **1 pts** Close (Three of four)

- **2 pts** Incorrect

2.5 2 / 2

✓ - **0 pts** **Correct (2^(n^2 - n) + 2^(n^2 +n / 2) - 2^(n^2 - n / 2))**

- **1 pts** Close

- **2 pts** Incorrect

QUESTION 3

Equivalence relation 10 pts

3.1 it is an equivalence relation 4 / 4

✓ - **0 pts** **Correct**

- **1 pts** issue in transitivity

- **3 pts** misunderstanding of what relation is saying

- **4 pts** blank

- **2 pts** misunderstanding of symmetry

- **1 pts** the decimal thing isn't exactly right, e.g. -.3 is related to .7

- **0 pts** Click here to replace this description.

- **1 pts** issue with symmetry

3.2 defining a function 2 / 4

- **0 pts** Correct

- **4 pts** blank

- **2 pts** need to prove uniqueness part of function

- **2 pts** missing existence part of function

✓ - **1 pts** **issue with uniqueness part of function**

✓ - 1 pts **need to consider different elements in the same equivalence class**

- 1 pts thing with decimals isn't quite right, for example -.3 and .7 are related

- 3 pts big misunderstanding of the equivalence relation or function

3.3 a function that doesn't descend 0 / 2

- 0 pts Correct

✓ - 2 pts **your g is not a function**

- 1 pts issue with justification

- 1 pts your g does not work

- 2 pts blank

QUESTION 4

m-ary tree 10 pts

4.1 number of internal vertices 5 / 5

✓ - 0 pts **Correct**

- 1 pts No/incorrect answer

- 4 pts No/incorrect justification

- 2 pts Didn't justify number of total vertices

- 3 pts "Proof by example"

- 2 pts Assumed every terminal vertex had the same height as the tree

- 5 pts Nothing

- 1 pts Forgot to account for root

- 2 pts Didn't subtract off internal vertices

4.2 height 3 / 5

- 0 pts Correct

- 1 pts No base case

- 1 pts Didn't set up/invoke induction

✓ - 1 pts **Backwards inductive step (didn't show inductive construction is exhaustive)**

- 2 pts Compared to complete tree without showing this case is extremal

- 3 pts Assumed tree is complete / inductive construction forms complete trees from complete trees

- 1 pts Assumed all immediate subtrees have height h-1

- 4 pts "Proof by example"

- 5 pts Nothing shown / Incorrect reasoning

- 1 **Point adjustment**

Define your variables!

Didn't state inductive hypothesis

QUESTION 5

spanning trees 10 pts

5.1 unique mst 3 / 6

- 0 pts Correct

✓ - 3 pts **Appeal to Prim's or Kruskal's Algorithm (without proving it can generate any MST)**

- 6 pts No / Invalid reasoning

5.2 non unique spanning tree 4 / 4

✓ - 0 pts **Correct**

- 4 pts Not an example

- 4 pts Claimed no such graph exists

- 4 pts Nothing

QUESTION 6

planar graphs 10 pts

6.1 2e > 3f 3 / 3

✓ + 3 pts **Correct**

+ 2 pts >= 3 edges for each face

+ 1 pts >= 3 edges for each face (w/ mistake)

+ 1 pts <=2 faces for each edge

+ 0 pts Incorrect

6.2 e<3v-6 3 / 3

✓ + 3 pts **Correct**

+ 2 pts Euler's formula

+ 1 pts Correct application with (a)

+ 0 pts Incorrect

6.3 nonplanar graph 3 / 4

+ 4 pts Correct

✓ + 3 pts **Isomorphic to K_3,3**

+ **2 pts** Mistaken/missing ismorphism to K_3,3

+ **1 pts** E <= 2v-4 or 2E >= 4F

+ **1 pts** Other partial credit

+ **0 pts** Incorrect

QUESTION 7

10 pts

7.1 7^n-1 divisible by 6 **5 / 5**

✓ + **5 pts** **Correct**

+ **1 pts** Base case

+ **1 pts** Inductive hypothesis

+ **2 pts** factoring out a 7 in inductive step as (6+1) or adding/substracting 7

+ **1 pts** Conclusion

+ **0 pts** Incorrect

7.2 number with only 1s divisible by 7 **0 / 5**

+ **5 pts** Correct

✓ + **0 pts** **Click here to replace this description.**

+ **1 pts** Look at 8 consecutive terms

+ **1 pts** Pigeonhole remainder

+ **1 pts** 7 divides a number of the form 111..000...

+ **2 pts** This implies that 7 divides 10^k*11...

+ **1 pts** Unsuccessful attempt with substantial work

QUESTION 8

balanced binary trees 10 pts

8.1 **4 / 4**

✓ - **0 pts** **Correct**

- **2 pts** incomplete, need to describe how a height n minimal balanced binary tree is made out of ones of smaller height

- **3 pts** can't just do examples

- **4 pts** blank

- **1 pts** how are you adding in these trees/ vertices?

- **3 pts** can't do induction without using some properties of minimal balanced binary trees

- **4 pts** incorrect numbers/ equation

8.2 relationship to fibonacci numbers **3 / 3**

✓ - **0 pts** **Correct**

- **1.5 pts** that is not the recurrence/ equation for the fibonacci numbers/ minimal balanced binary trees

- **1 pts** you are assuming the desired conclusion

- **3 pts** blank

- **1.5 pts** need to use recurrence for fiboacci numbers

- **1.5 pts** missing inductive step

- **1 pts** the two recurrences aren't exactly the same, you need to account for this difference

- **0.5 pts** error in equations

- **1 pts** need to check initial conditions

8.3 Theta **3 / 3**

✓ - **0 pts** **Correct**

- **0.5 pts** need to account for other term in equation for fibonacci numbers (sometimes it is contributing something positive, something something negative)

- **2 pts** wrong formula for fibonacci numbers/ v_n

- **1 pts** issue with big O

- **1 pts** issue with omega

- **3 pts** blank/ no gradable work

- **1 pts** wrong equations/ issues with constants

- **2 pts** need to use equation for v_n/ Fibonacci numbers

QUESTION 9

binomial coefficients 10 pts

9.1 3^n **4 / 4**

✓ + **4 pts** **Correct**

+ **3 pts** Minor error

+ **2 pts** Binomial theorem

+ **1 pts** Attempted induction or counting argument

+ **0 pts** Incorret

9.2 vandermonde identity **6 / 6**

✓ + **6 pts** **Correct**

+ **5 pts** Minor errror

+ **3 pts** One part of counting argument or (x+y)^n+m

+ **1 pts** Attempted to use induction/binomial thrm/Pascal's identity

+ **0 pts** Incorrect

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$

$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

   (a) $\frac{(n+k)!}{k!n!}$

   (b) $\frac{(n+1)!}{k!(n+1-k)!}$

   (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$

   (d) none of the above

$\frac{n!\,(k+1)}{k!\,(n-k)!\,(k+1)} + \frac{n!\,(n-k)}{(k+1)!\,(n-k-1)!\,(n-k)}$

$\frac{n!\,(n-\cancel{k}+\cancel{k}+1)}{(k+1)!\,(n-k)!} = \frac{(n+1)!}{(k+1)!\,(n-k)!}$

2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:

   (a) a height of $\geq \boxed{\lg(n!)}$

   (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)

   (c) a height of $O(\lg(n!))$

   (d) a height of $O(n \lg n)$

$|T(n)| \geq C\,|\lg(n!)|$

$h=3$

$n!=6$

$3 \geq \lg 6 \approx 2$

$\lg(n!) = \Theta(n \lg n)$, not necs.

$\lg(n!) \geq \Theta(n \log n)$

3. If G is a graph with n vertices and $n-2$ edges, then:

   (a) G is a tree

   (b) G is connected

   (c) G is disconnected

   (d) G is simple

$\lg(n) + \lg(n-1) + \lg(n-1)$

Question 1 continues on the next page...

Question 1 continued...

4. Which of these graphs has an Euler cycle? a every vertex has even degree.
   (a) K_4
   (b) K_5
   (c) $K_{3,3}$
   (d) $K_{2,3}$

5. What is the *fewest* number of $\boxed{\text{edges}}$ (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n. No edges for completely disconnected graph
   (a) Less than or equal to n
   (b) More than n but less than or equal to $n^2/2$
   (c) More than $n^2/2$ but less than or equal to n^2
   (d) More than n^2

$n \geq 0$

Best case

while ($z \in T$)
choose $v \in T$ with min $L(v)$
$T = T \setminus v$
for all neighbors u of v
$L(u) = \min(L(u), L(v) + w(v,u))$

$b+d=2$ $\quad b=2-d$ $\quad -5d=-5$ $\quad d=1$ $\quad 3(2-d)+2d=1$ $\quad b=2-d$

$3b-2d=1$ $\quad 3(2-d)-2d=1$ $\quad 5=d$ $\quad 6-3d+2d=1$ $\quad 3(2-d)+-2d=1$

$6-3d-2d=1$ $\quad -3=b$ $\quad 5=d$ $\quad 6-3d-2d=1$ $\quad -5d=-5$ $\quad d=1$ $\quad b=1$

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}?

$(t-3)(t+2)$

$t^n = t^{n-1} + 6t^{n-2}$ $\quad t^2 = t+6$ $\quad t^2-t-6=0$ $\quad (t-3)(t+2)=0$ $\quad b3^n + d(-2)^n = t^n$ $\quad b+d=2$ $\quad 3b+(-2)d=1$ $\quad d=1, b=1$

$\boxed{(1)(3^{100}) + (1)(2^{100})} = s_{100}$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

$7! = 6$

ABC CAB

7 spots for CS majors

$$\frac{7 \cdot P(7,4)}{11}$$

$S_n = S_{n-1}$

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

$$\binom{24}{4}$$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.

(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

$$\underbrace{2^{n^2-n}}_{\text{reflexive}} + \underbrace{2^{\frac{n^2-n}{2}+n}}_{\text{symmetric}} - \underbrace{2^{\frac{n^2-n}{2}}}_{\text{reflexive + symmetric}}$$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

Reflexive

For all $x \in \mathbb{R}$, $x - x = 0$, $0 \in \mathbb{Z}$, thus $(x, x) \in C$ for all $x \in \mathbb{R}$, C is reflexive.

Symmetric

If $(x, y) \in C$, $x - y = a \in \mathbb{Z}$. Then, $y - x = -a$. If $a \in \mathbb{Z}$, then $-a \in \mathbb{Z}$ (reflected over on the number line), so $y - x \in \mathbb{Z}$. $(y, x) \in C$, C is symmetric.

Transitive

If $(x, y), (y, z) \in C$, $x - y = a, y - z = b \in \mathbb{Z}$. $x - y + y - z = x - z \in \mathbb{Z}$ since $(x - y) + (y - z) = a + b \in \mathbb{Z}$ since a, b both $\in \mathbb{Z}$, so $(x, z) \in C$ and C is transitive.

C is equivalence relation for these 3 reasons.

Question 3 continues on the next page...

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.

Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

Every element in domain assigned to exactly 1 element in codomain

$f(x) = x + \frac{1}{2}$ gives 1 distinct result for all $x \in \mathbb{R}$, so it maps $[x]$ to $[x+\frac{1}{2}]$ for all $x \in \mathbb{R}$. Thus, every equivalence class $e = [x] \in \tilde{\mathbb{R}}$ will have exactly 1 result $e' = [x+\frac{1}{2}] \in \tilde{\mathbb{R}}$ in the codomain. In addition, all elements in the domain of $\tilde{\mathbb{R}}$ will be mapped since the domain of $f = \mathbb{R}$ covers entire domain of $\tilde{\mathbb{R}}$. In other words, for every $c \in \tilde{\mathbb{R}}$, there is a corresponding $x \in \mathbb{R}$, $x \in$ dom f such that $[x] = c$.

$f(x)$ is 1-1 since it passes vertical line test

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

~~If not all the elements in $\tilde{\mathbb{R}}$ can be mapped to another equivalence class, then $\tilde{g}$ is not a function.~~

for example $g(x) = \frac{1}{x}$. Here, there is $c = [0] \in \tilde{\mathbb{R}}$, but there is not the element $([0], [x]) \in \tilde{g}$ since $\frac{1}{0}$ is undefined. (there is no $[x]$ such that $[0] \tilde{g} [x]$).

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a ~~full~~ m-ary tree with i internal vertices, how many terminal vertices does T have?

Each vertex: m children except for root.

Total # vertices: $mi+1$ $\qquad$ $2i+1$

terminal vertices: $\boxed{(m-1)i+1}$ $\qquad$ $i+1$

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \leq m^h$.

Base Case ($h=0$)

If $h=0$, only 1 vertex, so $1 \leq m^0 = 1 \leq 1$, which is true

Inductive Case ($h=k$)

To form full m-ary tree of height k, we create a new root node and attach m children of height $k-1$

By inductive assumption

$t_{k-1} \leq m^{k-1}$

To show $t_k \leq m^k$, we add all the # of terminal nodes for all m children

of ~~nodes~~ terminal nodes for new tree (t_k) is $\leq m(m^{k-1}) = m^k$,

so $t_k \leq m^k$ □

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

Connected, weighted $\Rightarrow$ MST exists.

$G = (V, E)$ $|V| = n$

By Prim's algorithm, the MST of G will include the the lightest N edges in E. This forms a unique MST composed of these edges since there are no edges of the same weight, so there is only 1 edge choice for each of the n lightest weights. Each of these n lightest edges must be in the MST, since if they weren't, by contradiction we would have a MST T'.

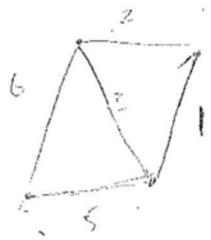
But we can replace an appropriately chosen edge in T' with one of the n lightest edges to form T, but now the weight of T < weight of T', which contradicts our ~~assumption~~ that T' is the MST of the graph G. $\square$

because weight of n lightest edges < weight of all other edges

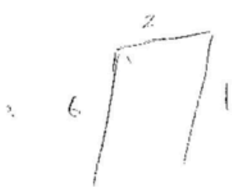
Question 5 continues on the next page...

(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.

Exists! :)



Edges distinct



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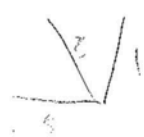

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6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

Each cycle is bounded by at least 3 edges,

so $E \geq 3F$. Then, some edges will be counted twice (belong to at most 2 cycles), so

$$E \geq \frac{3F}{2}.$$

$$2E \geq 3F.$$

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

Using previous result,

$$2E \geq 3F$$

With Euler's formula, we have

$$F = E - V + 2$$

$$2E \geq 3(E - V + 2)$$

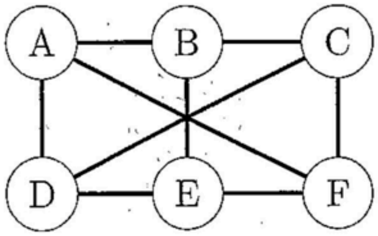
$$2E \geq 3E - 3V + 6$$

$$3V - 6 \geq E$$

$$E \leq 3V - 6 \quad \square$$

Question 6 continues on the next page...

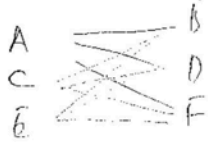
(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.



6 faces

$f = e - v + 2$

$9 - 6 + 2 = 5$



1. The graph contains a subgraph homeomorphic to $K_{3,3}$ (it is $K_{3,3}$), so it is **not** planar by Kuratowski's law.

$E \le 3V - 6$

$V = 6$
$E = 9$

Also, it doesn't have $9 - 6 + 2 = 5$ faces, which violates Euler's formula.

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Using induction,

Base Case ($i=1$)

$7^1 - 1 = 7 - 1 = 6$ $\Rightarrow 6/6 = 1$ ✓ (divisible by itself)

Inductive Case ($i = k+1$)

Assume $7^k - 1$ divisible by 6, show $7^{k+1} - 1$ divisible by 6

$7^{k+1} - 1 = 7(7^k) - 1$

$7(7^k - 1) - 1 + 7$

$= 7(7^k - 1) + 6$

$p = 7^k - 1$, p is divisible by 6 by inductive assumption

$7^{k+1} - 1 = 8p$, so $7^{k+1} - 1$ is multiple of ~~#~~ divisible by 6, so it is also divisible by 6 □

8421
11

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7. → prime, odd.

1010

~~$\frac{10^0 + 10^1 + 10^2 + 10^3 + \cdots 10^n}{7} = 0$~~

~~7 49 8 7 1~~

~~Show $\sum_{i=0}^{n} 10^i = 7k$, $k \in \mathbb{Z}$~~

1
11
111
1111
11111

$\sum_{i=0}^{n} 10^i = \sum_{i=0}^{n} (7+3)^i$

$= \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} \boxed{7^i} 3^{i-k}$

~~$= \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} 3^{i-k} \, 7^i$~~

Each term in the sum is divisible by at least 1 factor of 7,

~~$\sum_{i=0}^{n} 10^i 1^{n-1}$ $(10+1)^n$ $(7+3)^i$ $\sum 7^i 3^{k-i}$~~

~~Thus~~, we can factor one 7 out to obtain a number of the form $7k$, which is divisible by 7.

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$.

Base Case

$n=0 \quad V_n = 1$

$n=1 \quad V_n = 2$

$n=2 \quad V_n = 4$

Inductive Case

Assume V_k, is # of min vertices for balanced binary tree of height k, $k<n$

We can create new tree of height n by creating new root, then making 2 subtrees of height $(n-1)$ and $(n-2)$ because they can differ by at most 1. This gives us $V_n = V_{n-1} + V_{n-2} + 1$.

height of subtrees

root

(b) (3 points) Show that for $n \geq 0$, $v_n = F_{n+2}$, where F_k is the k^{th} Fibonacci number.

$V_n = F_{n+3} - 1$

$F_0\ F_1\ F_2\ F_3$

$0\ 1\ 1\ 2$

Base Case $(n=0)$

$V_0 = 1$

$V_0 = F_3 - 1 = 2 - 1 = 1$ ✓

Inductive Case $(n = k+1)$

Assume $V_k = F_{k+3} - 1$ for $0 \leq k < k+1$, show

$V_{k+1} = F_{k+4} - 1$

$V_{k+1} = V_k + V_{k-1} + 1$

$= (F_{k+3} - 1) + (F_{k+2} - \cancel{1}) \cancel{+1}$

$= F_{k+3} + F_{k+2} - 1$

$= F_{k+4} - 1$ by definition of Fibonacci sequence

✓

Question 8 continues on the next page...

$v_n = O(\phi^{n+2})$
$= \Omega(\phi^{n+2})$

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

(0)

$v_n = F_{n+2}$

$F_n = F_{n-1} + F_{n-2}$

$t^n = t^{n-1} + t^{n-2}$

$t^2 = t + 1$

$v_0 = 1$
$v_1 = 2$

$t^2 - t - 1 = 0$

$t = \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$v_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$

(1) For sufficiently large n, $\left(\frac{1-\sqrt{5}}{2}\right)^n = 0$ since $3 > \sqrt{5} > 2$

(2) since $v_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n$ for all but finitely many n,

$|v_n| \le c_1 |\phi^n|$
$|v_n| \ge c_2 |\phi^n|$ for all but finitely many positive n

(3) $v_n = \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$

(4) $\Theta(\phi^{n+2}) = \Theta(\phi^2 \phi^n) = \Theta(\phi^n)$ since ϕ^2 is just a constant that can be scaled up under

(5) since $v_n = \Theta(\phi^n)$, $\Theta(\phi^n) = \Theta(\phi^{n+2})$

then $v_n = \Theta(\phi^{n+2})$.

□

A

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

3^n = # ways to form 2 sets $A, B \subseteq X$, where $|X| = n$, and $A \subseteq B$

For each element in X, there are 3 choices:

① None include
② Both A and B $\Rightarrow 3^n$ possible ~~sets~~ subsets $A, B \subseteq N$, $A \subseteq B$
③ Just B

We can count all the possible subsets of X (i.e. # of B) by $\sum_{i=0}^{n} \binom{n}{i}$. However, for each of those subsets of size i, also have to form another subset (i.e. A), which has 2^i possibilities. Multiplied together, we have $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

$\binom{n+m}{r}$: # ways to choose r items from $n+m$ items (e.g. set X, Y $|X| = n$, $|Y| = m$)

For $i = 0 \ldots r$, we can choose i items from X
$r-i$ items from Y to

Since, $|X| = n$, $|Y| = m$, a total of r items.

There are $\binom{n}{i}\binom{m}{r-i}$ for each choice of i, and we can sum all of them ~~together~~ from i in $[0, r]$ together to obtain $\binom{n+m}{r}$ as the question states.

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turtually cool!