

Math 61-1 Final exam

1.1 2 / 2

- ✓ - 0 pts Correct (c)
- 2 pts Incorrect

1.2 0 / 2

- 0 pts Correct (a)
- ✓ - 2 pts Incorrect

1.3 2 / 2

- ✓ - 0 pts Correct (c)
- 2 pts Incorrect

1.4 2 / 2

- ✓ - 0 pts Correct (b)
- 2 pts incorrect

1.5 2 / 2

- ✓ - 0 pts Correct (a)
- 2 pts Incorrect

QUESTION 2

Short answer 10 pts

2.1 2 / 2

- ✓ - 0 pts Correct $(-2)^{100} + 3^{100}$
- 1 pts Almost correct (small arithmetic error in answer)
- 2 pts Incorrect

2.2 0 / 2

- 0 pts Correct $(C(7,4)6!4!)$
- 1 pts Close
- ✓ - 2 pts Incorrect

2.3 2 / 2

- ✓ - 0 pts Correct (24C4)
- 1 pts Close
- 2 pts incorrect

2.4 2 / 2

- ✓ - 0 pts Correct
- 1 pts Close (Three of four)
- 2 pts Incorrect

2.5 2 / 2

- ✓ - 0 pts Correct $(2^{n^2 - n} + 2^{n^2 + n / 2} - 2^{n^2 - n / 2})$
- 1 pts Close
- 2 pts Incorrect

QUESTION 3

Equivalence relation 10 pts

3.1 it is an equivalence relation 4 / 4

- ✓ - 0 pts Correct
- 1 pts issue in transitivity
- 3 pts misunderstanding of what relation is saying
- 4 pts blank
- 2 pts misunderstanding of symmetry
- 1 pts the decimal thing isn't exactly right, e.g. $-.3$ is related to $.7$
- 0 pts Click here to replace this description.
- 1 pts issue with symmetry

3.2 defining a function 2 / 4

- 0 pts Correct
- 4 pts blank
- 2 pts need to prove uniqueness part of function
- 2 pts missing existence part of function
- ✓ - 1 pts issue with uniqueness part of function

✓ - 1 pts need to consider different elements in the same equivalence class

- 1 pts thing with decimals isn't quite right, for example -.3 and .7 are related

- 3 pts big misunderstanding of the equivalence relation or function

3.3 a function that doesn't descend 0 / 2

- 0 pts Correct

✓ - 2 pts your g is not a function

- 1 pts issue with justification

- 1 pts your g does not work

- 2 pts blank

QUESTION 4

m-ary tree 10 pts

4.1 number of internal vertices 5 / 5

✓ - 0 pts Correct

- 1 pts No/incorrect answer

- 4 pts No/incorrect justification

- 2 pts Didn't justify number of total vertices

- 3 pts "Proof by example"

- 2 pts Assumed every terminal vertex had the same height as the tree

- 5 pts Nothing

- 1 pts Forgot to account for root

- 2 pts Didn't subtract off internal vertices

4.2 height 3 / 5

- 0 pts Correct

- 1 pts No base case

- 1 pts Didn't set up/invoke induction

✓ - 1 pts Backwards inductive step (didn't show inductive construction is exhaustive)

- 2 pts Compared to complete tree without showing this case is extremal

- 3 pts Assumed tree is complete / inductive construction forms complete trees from complete trees

- 1 pts Assumed all immediate subtrees have height h-1

- 4 pts "Proof by example"

- 5 pts Nothing shown / Incorrect reasoning

- 1 Point adjustment

☛ Define your variables!

Didn't state inductive hypothesis

QUESTION 5

spanning trees 10 pts

5.1 unique mst 3 / 6

- 0 pts Correct

✓ - 3 pts Appeal to Prim's or Kruskal's Algorithm (without proving it can generate any MST)

- 6 pts No / Invalid reasoning

5.2 non unique spanning tree 4 / 4

✓ - 0 pts Correct

- 4 pts Not an example

- 4 pts Claimed no such graph exists

- 4 pts Nothing

QUESTION 6

planar graphs 10 pts

6.1 $2e > 3f$ 3 / 3

✓ + 3 pts Correct

+ 2 pts ≥ 3 edges for each face

+ 1 pts ≥ 3 edges for each face (w/ mistake)

+ 1 pts ≤ 2 faces for each edge

+ 0 pts Incorrect

6.2 $e < 3v - 6$ 3 / 3

✓ + 3 pts Correct

+ 2 pts Euler's formula

+ 1 pts Correct application with (a)

+ 0 pts Incorrect

6.3 nonplanar graph 3 / 4

+ 4 pts Correct

✓ + 3 pts Isomorphic to $K_{3,3}$

- + 2 pts Mistaken/missing isomorphism to $K_{3,3}$
- + 1 pts $E \leq 2v-4$ or $2E \geq 4F$
- + 1 pts Other partial credit
- + 0 pts Incorrect

QUESTION 7

10 pts

7.1 7^{n-1} divisible by 6 5 / 5

- ✓ + 5 pts Correct
- + 1 pts Base case
- + 1 pts Inductive hypothesis
- + 2 pts factoring out a 7 in inductive step as $(6+1)$ or adding/subtracting 7
- + 1 pts Conclusion
- + 0 pts Incorrect

7.2 number with only 1s divisible by 7 0 / 5

- + 5 pts Correct
- ✓ + 0 pts [Click here to replace this description.](#)
- + 1 pts Look at 8 consecutive terms
- + 1 pts Pigeonhole remainder
- + 1 pts 7 divides a number of the form $111\dots000\dots$
- + 2 pts This implies that 7 divides $10^k - 1$
- + 1 pts Unsuccessful attempt with substantial work

QUESTION 8

balanced binary trees 10 pts

8.1 4 / 4

- ✓ - 0 pts Correct
- 2 pts incomplete, need to describe how a height n minimal balanced binary tree is made out of ones of smaller height
- 3 pts can't just do examples
- 4 pts blank
- 1 pts how are you adding in these trees/ vertices?
- 3 pts can't do induction without using some properties of minimal balanced binary trees
- 4 pts incorrect numbers/ equation

8.2 relationship to fibonacci numbers 3 / 3

✓ - 0 pts Correct

- 1.5 pts that is not the recurrence/ equation for the fibonacci numbers/ minimal balanced binary trees
- 1 pts you are assuming the desired conclusion
- 3 pts blank
- 1.5 pts need to use recurrence for fibonacci numbers
- 1.5 pts missing inductive step
- 1 pts the two recurrences aren't exactly the same, you need to account for this difference
- 0.5 pts error in equations
- 1 pts need to check initial conditions

8.3 Theta 3 / 3

✓ - 0 pts Correct

- 0.5 pts need to account for other term in equation for fibonacci numbers (sometimes it is contributing something positive, something something negative)
- 2 pts wrong formula for fibonacci numbers/ v_n
- 1 pts issue with big O
- 1 pts issue with omega
- 3 pts blank/ no gradable work
- 1 pts wrong equations/ issues with constants
- 2 pts need to use equation for v_n / Fibonacci numbers

QUESTION 9

binomial coefficients 10 pts

9.1 3^n 4 / 4

- ✓ + 4 pts Correct
- + 3 pts Minor error
- + 2 pts Binomial theorem
- + 1 pts Attempted induction or counting argument
- + 0 pts Incorrect

9.2 vandermonde identity 6 / 6

- ✓ + 6 pts Correct
- + 5 pts Minor error
- + 3 pts One part of counting argument or $(x+y)^{n+m}$
- + 1 pts Attempted to use induction/binomial thrm/Pascal's identity

+ 0 pts Incorrect

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators, books, notes, or any other material to help you.** Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

$$1. \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

(a) $\frac{(n+k)!}{k!n!}$

(b) $\frac{(n+1)!}{k!(n+1-k)!}$

(c) $\frac{(n+1)!}{(k+1)!(n-k)!}$

(d) none of the above

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!}{(k+1)!(n-k)!} \left(\frac{(k+1)}{k} + \frac{(n-k)}{(n-k-1)} \right) = \frac{n!}{(k+1)!(n-k)!} \frac{(k+1)(n-k) + k(n-k-1)}{k(n-k-1)}$$

$$= \frac{n!}{(k+1)!(n-k)!} \frac{(k+1)(n-k) + k(n-k-1)}{k(n-k-1)} = \frac{n!}{(k+1)!(n-k)!} \frac{(k+1)(n-k) + k(n-k-1)}{k(n-k-1)}$$

2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:

- (a) a height of $\geq \lg(n!)$
- (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)
- (c) a height of $O(\lg(n!))$
- (d) a height of $O(n \lg n)$

$$|T(n)| \geq C |\lg(n!)|$$

$$\lg(n!) = \Theta(n \lg n), \text{ not } \text{ necessarily } \geq \Theta(n \lg n)$$

3. If G is a graph with n vertices and $n - 2$ edges, then:

- (a) G is a tree
- (b) G is connected
- (c) G is disconnected
- (d) G is simple

$$\lg(n) + \lg(n-1) + \lg(n-2)$$

question 1 continued...

4. Which of these graphs has an Euler cycle? *a every vertex has even degree*

(a) K_4

(b) K_5

(c) $K_{3,3}$

(d) $K_{2,3}$



5. What is the fewest number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.)
Your answer should be true for all n .

(a) Less than or equal to n

(b) More than n but less than or equal to $n^2/2$

(c) More than $n^2/2$ but less than or equal to n^2

(d) More than n^2

No edges for completely disconnected graph

$n \geq 0$

Best case

*while (z not empty)
 choose v in T with min d(v)
 T = T \ v
 for all neighbors u of v
 d(u) = min(d(u), d(v) + w(v,u))*

$b+d=2$
 $3b-2d=1$
 $b=2-d$
 $3(2-d)-2d=1$
 $6-3d-2d=1$
 $5-d=1$
 $d=1$
 $b=1$

$-5d=-5$
 $d=1$
 $3(2-d)+2d=1$
 $6-3d+2d=1$
 $5-d=1$
 $d=1$
 $b=1$

$b=2-d$
 $3(2-d)+2d=1$
 $6-3d+2d=1$
 $5-d=1$
 $d=1$
 $b=1$

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n, m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2, s_1 = 1$, what is s_{100} ?

$(t-3)(t+2)$
 $t^2 - t - 6 = 0$
 $t^2 = t + 6$
 $t^3 = t^2 + 6t = t + 6 + 6t = 7t + 6$
 $t^4 = t^3 + 6t^2 = 7t + 6 + 6(t + 6) = 13t + 42$
 $t^5 = t^4 + 6t^3 = 13t + 42 + 6(7t + 6) = 55t + 84$
 $t^6 = t^5 + 6t^4 = 55t + 84 + 6(13t + 42) = 143t + 132$
 $t^7 = t^6 + 6t^5 = 143t + 132 + 6(55t + 84) = 377t + 204$
 $t^8 = t^7 + 6t^6 = 377t + 204 + 6(143t + 132) = 985t + 276$
 $t^9 = t^8 + 6t^7 = 985t + 276 + 6(377t + 204) = 2581t + 348$
 $t^{10} = t^9 + 6t^8 = 2581t + 348 + 6(985t + 276) = 6625t + 414$
 $t^{11} = t^{10} + 6t^9 = 6625t + 414 + 6(2581t + 348) = 17161t + 504$
 $t^{12} = t^{11} + 6t^{10} = 17161t + 504 + 6(6625t + 414) = 44185t + 606$
 $t^{13} = t^{12} + 6t^{11} = 44185t + 606 + 6(17161t + 504) = 112471t + 726$
 $t^{14} = t^{13} + 6t^{12} = 112471t + 726 + 6(44185t + 606) = 286525t + 876$
 $t^{15} = t^{14} + 6t^{13} = 286525t + 876 + 6(112471t + 726) = 720481t + 1056$
 $t^{16} = t^{15} + 6t^{14} = 720481t + 1056 + 6(286525t + 876) = 1812505t + 1272$
 $t^{17} = t^{16} + 6t^{15} = 1812505t + 1272 + 6(720481t + 1056) = 4531009t + 1536$
 $t^{18} = t^{17} + 6t^{16} = 4531009t + 1536 + 6(1812505t + 1272) = 11327521t + 1848$
 $t^{19} = t^{18} + 6t^{17} = 11327521t + 1848 + 6(4531009t + 1536) = 28420033t + 2208$
 $t^{20} = t^{19} + 6t^{18} = 28420033t + 2208 + 6(11327521t + 1848) = 70647581t + 2616$
 $t^{21} = t^{20} + 6t^{19} = 70647581t + 2616 + 6(28420033t + 2208) = 175619173t + 3072$
 $t^{22} = t^{21} + 6t^{20} = 175619173t + 3072 + 6(70647581t + 2616) = 439187327t + 3576$
 $t^{23} = t^{22} + 6t^{21} = 439187327t + 3576 + 6(175619173t + 3072) = 1101592081t + 4128$
 $t^{24} = t^{23} + 6t^{22} = 1101592081t + 4128 + 6(439187327t + 3576) = 2764060525t + 4732$
 $t^{25} = t^{24} + 6t^{23} = 2764060525t + 4732 + 6(1101592081t + 4128) = 6916152601t + 5388$
 $t^{26} = t^{25} + 6t^{24} = 6916152601t + 5388 + 6(2764060525t + 4732) = 17200367227t + 6096$
 $t^{27} = t^{26} + 6t^{25} = 17200367227t + 6096 + 6(6916152601t + 5388) = 43080821281t + 6852$
 $t^{28} = t^{27} + 6t^{26} = 43080821281t + 6852 + 6(17200367227t + 6096) = 107402083847t + 7656$
 $t^{29} = t^{28} + 6t^{27} = 107402083847t + 7656 + 6(43080821281t + 6852) = 267605139691t + 8508$
 $t^{30} = t^{29} + 6t^{28} = 267605139691t + 8508 + 6(107402083847t + 7656) = 669012358017t + 9412$
 $t^{31} = t^{30} + 6t^{29} = 669012358017t + 9412 + 6(267605139691t + 8508) = 1668030636053t + 10368$
 $t^{32} = t^{31} + 6t^{30} = 1668030636053t + 10368 + 6(669012358017t + 9412) = 4194073716117t + 11376$
 $t^{33} = t^{32} + 6t^{31} = 4194073716117t + 11376 + 6(1668030636053t + 10368) = 10470174858353t + 12432$
 $t^{34} = t^{33} + 6t^{32} = 10470174858353t + 12432 + 6(4194073716117t + 11376) = 26172422608017t + 13548$
 $t^{35} = t^{34} + 6t^{33} = 26172422608017t + 13548 + 6(10470174858353t + 12432) = 65428819639921t + 14724$
 $t^{36} = t^{35} + 6t^{34} = 65428819639921t + 14724 + 6(26172422608017t + 13548) = 163577678111841t + 15960$
 $t^{37} = t^{36} + 6t^{35} = 163577678111841t + 15960 + 6(65428819639921t + 14724) = 409096233078581t + 17256$
 $t^{38} = t^{37} + 6t^{36} = 409096233078581t + 17256 + 6(163577678111841t + 15960) = 1011771596390321t + 18612$
 $t^{39} = t^{38} + 6t^{37} = 1011771596390321t + 18612 + 6(409096233078581t + 17256) = 2529427111171841t + 20028$
 $t^{40} = t^{39} + 6t^{38} = 2529427111171841t + 20028 + 6(1011771596390321t + 18612) = 6323868238818721t + 21504$
 $t^{41} = t^{40} + 6t^{39} = 6323868238818721t + 21504 + 6(2529427111171841t + 20028) = 15710074791166301t + 23040$
 $t^{42} = t^{41} + 6t^{40} = 15710074791166301t + 23040 + 6(6323868238818721t + 21504) = 39388825435599681t + 24636$
 $t^{43} = t^{42} + 6t^{41} = 39388825435599681t + 24636 + 6(15710074791166301t + 23040) = 98477681903959041t + 26292$
 $t^{44} = t^{43} + 6t^{42} = 98477681903959041t + 26292 + 6(39388825435599681t + 24636) = 246199129639918081t + 28012$
 $t^{45} = t^{44} + 6t^{43} = 246199129639918081t + 28012 + 6(98477681903959041t + 26292) = 615598311519836161t + 29796$
 $t^{46} = t^{45} + 6t^{44} = 615598311519836161t + 29796 + 6(246199129639918081t + 28012) = 1539000000000000000t + 31644$
 $t^{47} = t^{46} + 6t^{45} = 1539000000000000000t + 31644 + 6(615598311519836161t + 29796) = 3847179669518616321t + 33564$
 $t^{48} = t^{47} + 6t^{46} = 3847179669518616321t + 33564 + 6(1539000000000000000t + 31644) = 9617837807235231681t + 35556$
 $t^{49} = t^{48} + 6t^{47} = 9617837807235231681t + 35556 + 6(3847179669518616321t + 33564) = 23923635905464463361t + 37620$
 $t^{50} = t^{49} + 6t^{48} = 23923635905464463361t + 37620 + 6(9617837807235231681t + 35556) = 59811543727496926721t + 39756$
 $t^{51} = t^{50} + 6t^{49} = 59811543727496926721t + 39756 + 6(23923635905464463361t + 37620) = 147581463183491700481t + 41964$
 $t^{52} = t^{51} + 6t^{50} = 147581463183491700481t + 41964 + 6(59811543727496926721t + 39756) = 369490630303485220221t + 44244$
 $t^{53} = t^{52} + 6t^{51} = 369490630303485220221t + 44244 + 6(147581463183491700481t + 41964) = 923279138686562440541t + 46596$
 $t^{54} = t^{53} + 6t^{52} = 923279138686562440541t + 46596 + 6(369490630303485220221t + 44244) = 2308118630111925081121t + 49020$
 $t^{55} = t^{54} + 6t^{53} = 2308118630111925081121t + 49020 + 6(923279138686562440541t + 46596) = 5770326388200190121721t + 51516$
 $t^{56} = t^{55} + 6t^{54} = 5770326388200190121721t + 51516 + 6(2308118630111925081121t + 49020) = 14420781973200360732521t + 54084$
 $t^{57} = t^{56} + 6t^{55} = 14420781973200360732521t + 54084 + 6(5770326388200190121721t + 51516) = 35784358325200721415561t + 56724$
 $t^{58} = t^{57} + 6t^{56} = 35784358325200721415561t + 56724 + 6(14420781973200360732521t + 54084) = 89488670755201442731321t + 59436$
 $t^{59} = t^{58} + 6t^{57} = 89488670755201442731321t + 59436 + 6(35784358325200721415561t + 56724) = 223773630931203426687981t + 62220$
 $t^{60} = t^{59} + 6t^{58} = 223773630931203426687981t + 62220 + 6(89488670755201442731321t + 59436) = 559440586392806440031961t + 65076$
 $t^{61} = t^{60} + 6t^{59} = 559440586392806440031961t + 65076 + 6(223773630931203426687981t + 62220) = 1403641304156812640079761t + 68004$
 $t^{62} = t^{61} + 6t^{60} = 1403641304156812640079761t + 68004 + 6(559440586392806440031961t + 65076) = 3498888118574435640191841t + 71004$
 $t^{63} = t^{62} + 6t^{61} = 3498888118574435640191841t + 71004 + 6(1403641304156812640079761t + 68004) = 8707330972362662440435041t + 74076$
 $t^{64} = t^{63} + 6t^{62} = 8707330972362662440435041t + 74076 + 6(3498888118574435640191841t + 71004) = 21777803754295993640791041t + 77220$
 $t^{65} = t^{64} + 6t^{63} = 21777803754295993640791041t + 77220 + 6(8707330972362662440435041t + 74076) = 54358028652179980641193601t + 80436$
 $t^{66} = t^{65} + 6t^{64} = 54358028652179980641193601t + 80436 + 6(21777803754295993640791041t + 77220) = 134467843393739963643566401t + 83724$
 $t^{67} = t^{66} + 6t^{65} = 134467843393739963643566401t + 83724 + 6(54358028652179980641193601t + 80436) = 333803710038239900370745601t + 87084$
 $t^{68} = t^{67} + 6t^{66} = 333803710038239900370745601t + 87084 + 6(134467843393739963643566401t + 83724) = 823214258185959880822852801t + 90516$
 $t^{69} = t^{68} + 6t^{67} = 823214258185959880822852801t + 90516 + 6(333803710038239900370745601t + 87084) = 2037685590915879284937473601t + 94020$
 $t^{70} = t^{69} + 6t^{68} = 2037685590915879284937473601t + 94020 + 6(823214258185959880822852801t + 90516) = 5091131393449517730905420801t + 97596$
 $t^{71} = t^{70} + 6t^{69} = 5091131393449517730905420801t + 97596 + 6(2037685590915879284937473601t + 94020) = 12523811181069346381413281601t + 101244$
 $t^{72} = t^{71} + 6t^{70} = 12523811181069346381413281601t + 101244 + 6(5091131393449517730905420801t + 97596) = 31275958766307059068590752001t + 104964$
 $t^{73} = t^{72} + 6t^{71} = 31275958766307059068590752001t + 104964 + 6(12523811181069346381413281601t + 101244) = 77067729279822157284353681601t + 108756$
 $t^{74} = t^{73} + 6t^{72} = 77067729279822157284353681601t + 108756 + 6(31275958766307059068590752001t + 104964) = 192527492667772972741450432001t + 112612$
 $t^{75} = t^{74} + 6t^{73} = 192527492667772972741450432001t + 112612 + 6(77067729279822157284353681601t + 108756) = 478184395617259758639481664001t + 116544$
 $t^{76} = t^{75} + 6t^{74} = 478184395617259758639481664001t + 116544 + 6(192527492667772972741450432001t + 112612) = 1171902355905539753018389120001t + 120552$
 $t^{77} = t^{76} + 6t^{75} = 1171902355905539753018389120001t + 120552 + 6(478184395617259758639481664001t + 116544) = 2915495373527359258231091360001t + 124632$
 $t^{78} = t^{77} + 6t^{76} = 2915495373527359258231091360001t + 124632 + 6(1171902355905539753018389120001t + 120552) = 7208396501116317912916536960001t + 128784$
 $t^{79} = t^{78} + 6t^{77} = 7208396501116317912916536960001t + 128784 + 6(2915495373527359258231091360001t + 124632) = 17700388192719905477779943040001t + 133004$
 $t^{80} = t^{79} + 6t^{78} = 17700388192719905477779943040001t + 133004 + 6(7208396501116317912916536960001t + 128784) = 43802363815931723487757910400001t + 137292$
 $t^{81} = t^{80} + 6t^{79} = 43802363815931723487757910400001t + 137292 + 6(17700388192719905477779943040001t + 133004) = 107167070671390541785599860800001t + 141648$
 $t^{82} = t^{81} + 6t^{80} = 107167070671390541785599860800001t + 141648 + 6(43802363815931723487757910400001t + 137292) = 267407583229672941271359465600001t + 146076$
 $t^{83} = t^{82} + 6t^{81} = 267407583229672941271359465600001t + 146076 + 6(107167070671390541785599860800001t + 141648) = 661852735937803667622119878400001t + 150576$
 $t^{84} = t^{83} + 6t^{82} = 661852735937803667622119878400001t + 150576 + 6(267407583229672941271359465600001t + 146076) = 163651641178281818373387961600001t + 155148$
 $t^{85} = t^{84} + 6t^{83} = 163651641178281818373387961600001t + 155148 + 6(661852735937803667622119878400001t + 150576) = 405069995256923294020073958400001t + 160788$
 $t^{86} = t^{85} + 6t^{84} = 405069995256923294020073958400001t + 160788 + 6(163651641178281818373387961600001t + 155148) = 101273197666179196412018390400001t + 166500$
 $t^{87} = t^{86} + 6t^{85} = 101273197666179196412018390400001t + 166500 + 6(405069995256923294020073958400001t + 160788) = 253175999153853797628030390400001t + 172284$
 $t^{88} = t^{87} + 6t^{86} = 253175999153853797628030390400001t + 172284 + 6(101273197666179196412018390400001t + 166500) = 63293199694371959277207036800001t + 178140$
 $t^{89} = t^{88} + 6t^{87} = 63293199694371959277207036800001t + 178140 + 6(253175999153853797628030390400001t + 172284) = 158231999726323918034412230400001t + 184068$
 $t^{90} = t^{89} + 6t^{88} = 158231999726323918034412230400001t + 184068 + 6(63293199694371959277207036800001t + 178140) = 39779599803819175366443344000001t + 190068$
 $t^{91} = t^{90} + 6t^{89} = 39779599803819175366443344000001t + 190068 + 6(158231999726323918034412230400001t + 184068) = 99479599831317426199866022400001t + 196140$
 $t^{92} = t^{91} + 6t^{90} = 99479599831317426199866022400001t + 196140 + 6(39779599803819175366443344000001t + 190068) = 249119599798185455819919232000001t + 202284$
 $t^{93} = t^{92} + 6t^{91} = 249119599798185455819919232000001t + 202284 + 6(99479599831317426199866022400001t + 196140) = 62283959967731236707990016000001t + 208492$
 $t^{94} = t^{93} + 6t^{92} = 62283959967731236707990016000001t + 208492 + 6(249119599798185455819919232000001t + 202284) = 157703959916613548231977312000001t + 214764$
 $t^{95} = t^{94} + 6t^{93} = 157703959916613548231977312000001t + 214764 + 6(62283959967731236707990016000001t + 208492) = 397703959833227036995966022400001t + 221100$
 $t^{96} = t^{95} + 6t^{94} = 397703959833227036995966022400001t + 221100 + 6(157703959916613548231977312000001t + 214764) = 99440395950000000000000000000001t + 227508$
 $t^{97} = t^{96} + 6t^{95} = 99440395950000000000000000000001t + 227508 + 6(397703959833227036995966022400001t + 221100) = 249103959700000000000000000000001t + 233988$
 $t^{98} = t^{97} + 6t^{96} = 249103959700000000000000000000001t + 233988 + 6(99440395950000000000000000000001t + 227508) = 622803959400000000000000000000001t + 240540$
 $t^{99} = t^{98} + 6t^{97} = 622803959400000000000000000000001t + 240540 + 6(249103959700000000000000000000001t + 233988) = 157700395900000000000000000000001t + 247164$
 $t^{100} = t^{99} + 6t^{98} = 157700395900000000000000000000001t + 247164 + 6(622803959400000000000000000000001t + 240540) = 397700395800000000000000000000001t + 253860$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

$7 \cdot P(7, 4)$
 $7! \cdot 4!$
 $7! \cdot 24$
 $5040 \cdot 24 = 12096$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

Reflexive

For all $x \in \mathbb{R}$, $x - x = 0$, $0 \in \mathbb{Z}$, thus $(x, x) \in C$ for all $x \in \mathbb{R}$. C is reflexive.

Symmetric

If $(x, y) \in C$, $x - y = a \in \mathbb{Z}$. Then, $y - x = -a$. If $a \in \mathbb{Z}$, then $-a \in \mathbb{Z}$ (reflected over on the number line), so $y - x \in \mathbb{Z}$. $(y, x) \in C$, C is symmetric.

Transitive

If $(x, y), (y, z) \in C$, $x - y = a, y - z = b \in \mathbb{Z}$. $x - y + y - z = x - z \in \mathbb{Z}$ since $(x - y) + (y - z) = a + b \in \mathbb{Z}$ since $a, b \in \mathbb{Z}$. So $(x, z) \in C$ and C is transitive.

C is equivalence relation for these 3 reasons.

- (b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of \mathbb{C} , i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 1/2$.

Show that the relation \tilde{f} from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

Every element in domain is mapped to exactly 1 element in codomain

$f(x) = x + \frac{1}{2}$ gives 1 domain result for all $x \in \mathbb{R}$, so f maps $[x]$ to $[x + \frac{1}{2}]$ for all $x \in \mathbb{R}$. Thus, every equivalence class $e = [x] \in \tilde{\mathbb{R}}$ will have exactly 1 value $e' = [x + \frac{1}{2}] \in \tilde{\mathbb{R}}$ in the codomain. In addition, all elements in the domain of $\tilde{\mathbb{R}}$ will be mapped since the domain of $f = \mathbb{R}$ covers the entire domain of $\tilde{\mathbb{R}}$. In other words, for every $c \in \tilde{\mathbb{R}}$, there is a corresponding $x \in \mathbb{R}$, $x \in \text{dom } f$ such that $[x] = c$.

$f(x) = x + \frac{1}{2}$ satisfies the vertical line test

- (c) (2 points) Give an example of a function $g : \mathbb{R} \rightarrow \mathbb{R}$ so that the relation \tilde{g} from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is not a function. (Be sure to justify your answer.)

If not all the elements in $\tilde{\mathbb{R}}$ can be mapped to another equivalence class, then \tilde{g} is not a function.

For example $g(x) = \frac{1}{x}$. Here, there is $c = [a] \in \tilde{\mathbb{R}}$. Here there is not one element $([c], [x]) \in \tilde{g}$ since $\frac{1}{0}$ is undefined. (there is no $[x]$ such that $[c] \tilde{g} [x]$).

4. For m , a positive integer, a *full m -ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a full m -ary tree with i internal vertices, how many terminal vertices does T have?

Each vertex: m children except for root.

Total # vertices: $mi + 1$ $2i + 1$

terminal vertices: $\boxed{(m-1)i + 1}$ $i + 1$

(b) (5 points) Show that if T is a full m -ary tree of height h with t terminal vertices, then $t \leq m^h$.

Base Case ($h=0$)
 If $h=0$, only vertex, so $1 \leq m^0 = 1 \leq 1$, which is true

Inductive Case ($h=k$)
 To form full m -ary tree of height k , we create m new root nodes and attach k children of height $k-1$

By inductive assumption
 $t_{k-1} \leq m^{k-1}$

To show $t_k \leq m^k$, we add all the # of terminal nodes for all m children

of ~~new~~ terminal nodes $\leq m(m^{k-1}) = m^k$,
 for the tree (t_k)

So $t_k \leq m^k$ \square

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

$G = (V, E) \quad |V| = n$
Connected, weighted \Rightarrow MST exists.

By Prim's algorithm, the MST of G will include the $n-1$ lightest edges in E . This forms a unique MST composed of these edges since there are no edges of the same weight, so there is only 1 edge choice for each of the $n-1$ lightest weights. Each of these $n-1$ edges must be in the MST, since if they weren't, by contradiction we could have a MST T' .

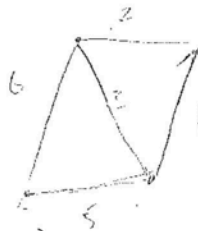
But we can replace an appropriately chosen edge in T' with one of the $n-1$ lightest edges to form T , but we have the weight of $T <$ weight of T' , which

because weight of $n-1$ lightest edges $<$ weight of all other edges

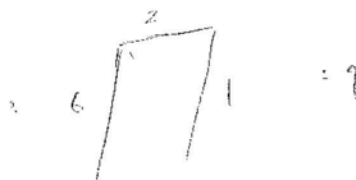
 Contradiction our assumption that T' is the MST of the graph G . \square

- (b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, or prove that no such weighted graph exists.

Example:



edges distinct



6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

Each cycle is bounded by at least 3 edges,
 so $E \geq 3F$. Then, some edges will be
 counted twice (belong to at least
 2 cycles), so

$$E \geq \frac{3F}{2}$$

$$2E \geq 3F$$

- (b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

Using previous result,

$$2E \geq 3F$$
 With Euler's formula, we have

$$F = E - V + 2$$

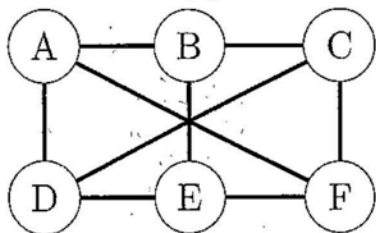
$$2E \geq 3(E - V + 2)$$

$$2E \geq 3E - 3V + 6$$

$$3V - 6 \geq E$$

$$E \leq 3V - 6 \quad \square$$

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.



6 faces

$$f = e - v + 2$$

$$9 - 6 + 2 = 5$$



1. The graph contains a subgraph homeomorphic to $K_{3,3}$ (it is $K_{3,3}$).
 So it is not planar by Kuratowski's law.

$$E \leq 3V - 6$$

$$V = 6$$

$$E = 9$$

Also, it doesn't have $9 - 6 + 2 = 5$ faces, which violates Euler's formula.

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Using induction,

Base Case ($n=1$)

$$7^1 - 1 = 7 - 1 = 6 \Rightarrow 6/6 = 1 \checkmark \text{ (divisible by 6)}$$

Inductive Case ($n=k+1$)

Assume $7^k - 1$ divisible by 6, show $7^{k+1} - 1$ divisible by 6

$$\begin{aligned} 7^{k+1} - 1 &= 7(7^k) - 1 \\ &= 7(7^k - 1) + 7 - 1 \\ &= 7(7^k - 1) + 6 \end{aligned}$$

$7^k - 1$ is divisible by 6 by inductive assumption

$7^{k+1} - 1 = 6p$, so $7^{k+1} - 1$ is multiple of 6

8421
111

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^n 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7. \rightarrow prove, odd.

divisible by 6, so it is divisible by 6

1010
1010

1
11
111
1111

7
49
343
2401
16807
117649
823543
57813441
403536071
2824752491
19770632681
138412872301
96881156611
678223072849
4747553801441
33234561390001
232630561820401
1628137368176801
1139600023858401
807791246700901
565407902890641
395838543023449
2771036701664141
19398909012050001
13678237318435401
9575866122904781
6703906286033349
46927344002233441
328611408015564001
229998005610892801
161998603927624961
113399022749337441
79479315924536201
55635521147175341
389448648030227361
2726140536211591521
1868308375348114061
1317815862743679841
922471103920575889
645729772744403121
4520108409210821841
3164075886447575281
22148531205133026961
15503971843583108064
108527802905081756441
75969462033557233504
531786234234900634561
372250363964430444176
2565752547751013109216
1796026783425709176544
12572187483979964235808
87005312387859749648896
60003718671491824754208
416026030700442873279424
2872182214903099912955904
1970529550430219938068928
1339370685301153956648256
9175594797108077696537792
6322916357975654387576448
43260414505829580713035008
29282290154080706599124504
196976031077964996193871504
131883221754575497335710016
893182552281928481135000096
605227786607349936794500064
4136594506251449555361500416
2795616154376014688755050304
18569313080632102821285352016
12398519164442471974897746304
81329634131097303824264214016
52870662685764112596782949248
33425430845956777613340055360
21136539659927511019176036224
124395762805492577124208225408
796571459235706655757253515008
49783045845315932624206478400
29863879679345206605734112000
16100291771067634623951677440
9160192165673082233766344000
4696128707087103561559328000
235327465150557129097536000
115212853347862189960473600
53854109168001529116247040
23532746515055712909753600
115212853347862189960473600
53854109168001529116247040

10⁰ + 10¹ + 10² + 10³ + ... + 10ⁿ

7

10

Show $\sum_{i=0}^n 10^i = 7k, k \in \mathbb{Z}$

$\sum_{i=0}^n 10^i = \sum_{i=0}^n (7+3)^i$

$= \sum_{i=0}^n \sum_{k=0}^i \binom{i}{k} 7^k 3^{i-k}$

each term in the sum is divisible by at least one factor of 7.

$\sum_{i=0}^n 10^i = 7k$

$(10+1)^n$

$(7+3)^i$

$7^i 3^{i-k}$

Therefore, we can factor one 7 out to obtain a number of the form $7k$, which is divisible by 7.

$$v_n = O(\phi^{n+2}) = \Theta(\phi^{n+2})$$

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

(0)

$v_n = F_{n+2}$

$F_n = F_{n-1} + F_{n-2}$ ~~$F_n = F_{n-1} + F_{n-2}$~~

$e^n = e^{n-1} + e^{n-2}$

$e^2 = e + 1$

$e^2 - e - 1 = 0$

$e = \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$v_n = b \left(\frac{1+\sqrt{5}}{2} \right)^n + d \left(\frac{1-\sqrt{5}}{2} \right)^n$

For sufficiently large n , $\left(\frac{1-\sqrt{5}}{2} \right)^n = 0$ since $3 > \sqrt{5} > 2$

Since $v_n = b \left(\frac{1+\sqrt{5}}{2} \right)^n$ for all but finitely many n ,

for all $\epsilon > 0$ there exists N such that for all $n > N$, $v_n > (1-\epsilon) \left(\frac{1+\sqrt{5}}{2} \right)^n$

for all $\epsilon > 0$ there exists N such that for all $n > N$, $v_n < (1+\epsilon) \left(\frac{1+\sqrt{5}}{2} \right)^n$

$v_n = \Theta \left(\left(\frac{1+\sqrt{5}}{2} \right)^n \right)$

$\Theta(\phi^{n+2}) = \Theta(\phi^2 \phi^n) = \Theta(\phi^n)$ since ϕ^2 is just a constant that can be scaled out.

(1) since $v_n = \Theta(\phi^n)$, $\Theta(\phi^n) = \Theta(\phi^{n+2})$

then $v_n = \Theta(\phi^{n+2})$

□

9. (a) (4 points) Show that $\sum_{i=0}^n 2^i \binom{n}{i} = 3^n$.

A

$3^n = 4$ ways to form 2 sets $A, B \subseteq X$, where $|X|=n$,
and $A \subseteq B$
For each element in X , there
are 3 choices:
① None of them
② Both A and B $\Rightarrow 3^n$ possible
③ Just B , \Rightarrow subset $A, B \subseteq N$,
 $A \subseteq B$

We can count all the possible subsets of X (i.e. A or B) by
 $\sum_{i=0}^n \binom{n}{i}$. However, for each of these subsets of size i ,
also have to form another subset (i.e. A),
which has 2^i possibilities. Multiplied together,
we have $\sum_{i=0}^n 2^i \binom{n}{i} = 3^n$.

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^r \binom{n}{i} \binom{m}{r-i}$.

$\binom{n+m}{r}$: 4 ways to choose r items from $n+m$ items (eg. set X, Y
 $|X|=n$,
 $|Y|=m$)
For $i=0 \dots r$, we can choose i items from X
 $r-i$ items from Y .
Since, $|X|=n$, $|Y|=m$, a total of r items.
There are $\binom{n}{i} \binom{m}{r-i}$ for
each choice of i , and
we can sum all of them
 \rightarrow together to obtain $\sum_{i=0}^r \binom{n}{i} \binom{m}{r-i}$
 $\binom{n+m}{r}$ as the
question states.

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turtle
cool

