Math 61-1 Final exam

BENJAMIN HE

TOTAL POINTS

55 / 90

QUESTION 1

Multiple choice 10 pts

1.1 **2 / 2**

- ✓ **- 0 pts** **Correct (c)**
- **- 2 pts** Incorrect

1.2 **0 / 2**

- **- 0 pts** Correct (a)
- ✓ **- 2 pts** **Incorrect**

1.3 **2 / 2**

- ✓ **- 0 pts** **Correct (c)**
- **- 2 pts** Incorrect

1.4 **2 / 2**

- ✓ **- 0 pts** **Correct (b)**
- **- 2 pts** incorrect

1.5 **0 / 2**

- **- 0 pts** Correct (a)
- ✓ **- 2 pts** **Incorrect**

QUESTION 2

Short answer 10 pts

2.1 **2 / 2**

- ✓ **- 0 pts** **Correct ((-2)^100 + 3^100)**
- **- 1 pts** Almost correct (small arithmetic error in answer)
- **- 2 pts** Incorrect

2.2 **2 / 2**

- ✓ **- 0 pts** **Correct (C(7,4)6!4!)**
- **- 1 pts** Close
- **- 2 pts** Incorrect

2.3 **2 / 2**

- ✓ **- 0 pts** **Correct (24C4)**
- **- 1 pts** Close
- **- 2 pts** incorrect

2.4 **0 / 2**

- **- 0 pts** Correct
- **- 1 pts** Close (Three of four)
- ✓ **- 2 pts** **Incorrect**

2.5 **1 / 2**

- **- 0 pts** Correct (2^(n^2 - n) + 2^(n^2 +n / 2) - 2^(n^2 - n / 2))
- ✓ **- 1 pts** **Close**
- **- 2 pts** Incorrect

QUESTION 3

Equivalence relation 10 pts

3.1 it is an equivalence relation **4 / 4**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** issue in transitivity
- **- 3 pts** misunderstanding of what relation is saying
- **- 4 pts** blank
- **- 2 pts** misunderstanding of symmetry
- **- 1 pts** the decimal thing isn't exactly right, e.g. -.3 is related to .7
- **- 0 pts** Click here to replace this description.
- **- 1 pts** issue with symmetry

3.2 defining a function **3 / 4**

- **- 0 pts** Correct
- **- 4 pts** blank
- **- 2 pts** need to prove uniqueness part of function
- **- 2 pts** missing existence part of function
- **- 1 pts** issue with uniqueness part of function

✓ **- 1 pts need to consider different elements in the same equivalence class**

- 1 pts thing with decimals isn't quite right, for example -.3 and .7 are related

- 3 pts big misunderstanding of the equivalence relation or function

3.3 a function that doesn't descend 1 / 2

- 0 pts Correct

- 2 pts your g is not a function

✓ **- 1 pts issue with justification**

- 1 pts your g does not work

- 2 pts blank

why?

QUESTION 4

m-ary tree 10 pts

4.1 number of internal vertices 0 / 5

- 0 pts Correct

✓ **- 1 pts No/incorrect answer**

✓ **- 4 pts No/incorrect justification**

- 2 pts Didn't justify number of total vertices

- 3 pts "Proof by example"

- 2 pts Assumed every terminal vertex had the same height as the tree

- 5 pts Nothing

- 1 pts Forgot to account for root

- 2 pts Didn't subtract off internal vertices

4.2 height 1.5 / 5

- 0 pts Correct

- 1 pts No base case

- 1 pts Didn't set up/invoke induction

- 1 pts Backwards inductive step (didn't show inductive construction is exhaustive)

- 2 pts Compared to complete tree without showing this case is extremal

✓ **- 3 pts Assumed tree is complete / inductive construction forms complete trees from complete trees**

- 1 pts Assumed all immediate subtrees have height h-1

- 4 pts "Proof by example"

- 5 pts Nothing shown / Incorrect reasoning

- 0.5 Point adjustment

Your t and h mean two different things each! This is bad.

QUESTION 5

spanning trees 10 pts

5.1 unique mst 3 / 6

- 0 pts Correct

- 3 pts Appeal to Prim's or Kruskal's Algorithm (without proving it can generate any MST)

- 6 pts No / Invalid reasoning

- 3 Point adjustment

What if there are MST's which differ by more than a single pair of edges?

5.2 non unique spanning tree 4 / 4

✓ **- 0 pts Correct**

- 4 pts Not an example

- 4 pts Claimed no such graph exists

- 4 pts Nothing

QUESTION 6

planar graphs 10 pts

6.1 2e > 3f 3 / 3

✓ **+ 3 pts Correct**

+ 2 pts >= 3 edges for each face

+ 1 pts >= 3 edges for each face (w/ mistake)

+ 1 pts <=2 faces for each edge

+ 0 pts Incorrect

6.2 e<3v-6 3 / 3

✓ **+ 3 pts Correct**

+ 2 pts Euler's formula

+ 1 pts Correct application with (a)

+ 0 pts Incorrect

6.3 nonplanar graph **1 / 4**

+ **4 pts** Correct

+ **3 pts** Isomorphic to K_3,3

+ **2 pts** Mistaken/missing ismorphism to K_3,3

+ **1 pts** E <= 2v-4 or 2E >= 4F

✓ + **1 pts** **Other partial credit**

+ **0 pts** Incorrect

QUESTION 7

10 pts

7.1 7^n-1 divisible by 6 **5 / 5**

✓ + **5 pts** **Correct**

+ **1 pts** Base case

+ **1 pts** Inductive hypothesis

+ **2 pts** factoring out a 7 in inductive step as (6+1) or adding/substracting 7

+ **1 pts** Conclusion

+ **0 pts** Incorrect

7.2 number with only 1s divisible by 7 **0 / 5**

+ **5 pts** Correct

✓ + **0 pts** **Click here to replace this description.**

+ **1 pts** Look at 8 consecutive terms

+ **1 pts** Pigeonhole remainder

+ **1 pts** 7 divides a number of the form 111..000...

+ **2 pts** This implies that 7 divides 10^k*11...

+ **1 pts** Unsuccessful attempt with substantial work

QUESTION 8

balanced binary trees 10 pts

8.1 **4 / 4**

✓ - **0 pts** **Correct**

- **2 pts** incomplete, need to describe how a height n minimal balanced binary tree is made out of ones of smaller height

- **3 pts** can't just do examples

- **4 pts** blank

- **1 pts** how are you adding in these trees/ vertices?

- **3 pts** can't do induction without using some properties of minimal balanced binary trees

- **4 pts** incorrect numbers/ equation

8.2 relationship to fibonacci numbers **3 / 3**

✓ - **0 pts** **Correct**

- **1.5 pts** that is not the recurrence/ equation for the fibonacci numbers/ minimal balanced binary trees

- **1 pts** you are assuming the desired conclusion

- **3 pts** blank

- **1.5 pts** need to use recurrence for fiboacci numbers

- **1.5 pts** missing inductive step

- **1 pts** the two recurrences aren't exactly the same, you need to account for this difference

- **0.5 pts** error in equations

- **1 pts** need to check initial conditions

8.3 Theta **2.5 / 3**

- **0 pts** Correct

✓ - **0.5 pts** **need to account for other term in equation for fibonacci numbers (sometimes it is contributing something positive, something something negative)**

- **2 pts** wrong formula for fibonacci numbers/ v_n

- **1 pts** issue with big O

- **1 pts** issue with omega

- **3 pts** blank/ no gradable work

- **1 pts** wrong equations/ issues with constants

- **2 pts** need to use equation for v_n/ Fibonacci numbers

QUESTION 9

binomial coefficients 10 pts

9.1 3^n **4 / 4**

✓ + **4 pts** **Correct**

+ **3 pts** Minor error

+ **2 pts** Binomial theorem

+ **1 pts** Attempted induction or counting argument

+ **0 pts** Incorret

9.2 vandermonde identity **0 / 6**

+ **6 pts** Correct

+ **5 pts** Minor errror

+ **3 pts** One part of counting argument or (x+y)^n+m

+ **1 pts** Attempted to use induction/binomial thrm/Pascal's identity

✓ + **0 pts** **Incorrect**

J. Cameron
Math 61
Friday, March 22nd, 2019

Final

Name: Benjamin He

Student ID: 804962948

Section:

$=\binom{n}{j+1}+\binom{n-1}{j+1}(?)$

$(2n+1)C_{n-1}$

$n+2$

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	[✓ 1E]	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$
   - (a) $\frac{(n+k)!}{k!n!}$
   - (b) $\frac{(n+1)!}{k!(n+1-k)!}$
   - [✓ (c)] $\frac{(n+1)!}{(k+1)!(n-k)!}$
   - (d) none of the above

n!

$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

$n+\cancel{1}-(k\cancel{+1})$

2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:
   - (a) a height of $\geq \lg(n!)$
   - (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)
   - (c) a height of $O(\lg(n!))$
   - [✓ (d)] a height of $O(n \lg n)$

3. If G is a graph with n vertices and $n-2$ edges, then:
   - (a) G is a tree
   - (b) G is connected
   - [✓ (c)] G is disconnected
   - (d) G is simple

Question 1 continues on the next page...

Question 1 continued. . .

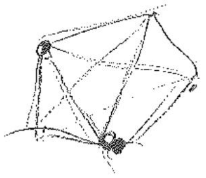
4. Which of these graphs has an Euler cycle?
   - (a) K_4
   - [✓ (b) K_5]
   - (c) $K_{3,3}$
   - (d) $K_{2,3}$

connected
all degrees even

⑤ What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.
   - ~~(a)~~ Less than or equal to n
   - (b) More than n but less than or equal to $n^2/2$
   - [✓ (c) More than $n^2/2$ but less than or equal to n^2]
   - ~~(d)~~ More than n^2

4 13

$n(n-1)$



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2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}?

$s_n = t^n$

$t^n = t^{n-1} + 6t^{n-2} \rightarrow t^2 - t - 6 = 0$

$(t-3)(t+2) = 0 \rightarrow t = 3, t = -2$

$s_n = a3^n + b(-2)^n$ $a = 1, b = 1$

$a + b = 2$ $s_n = 3^n + (-2)^n$

$3a - 2b = 1$ $\boxed{s_{100} = 3^{100} + (-2)^{100}}$

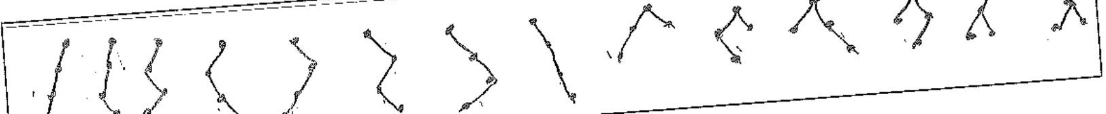
(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

$\boxed{6!\,P(7,4)}$

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

$\boxed{\binom{24}{4} = \binom{24}{20}}$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.



(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

$2^{\frac{1}{2}(n^2-n)}$ $2^n - 2^{\frac{1}{2}(n^2-n)} = \boxed{2^{\frac{1}{2}(n^2-n)}(2^n - 1)}$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \overset{\text{Real}}{\mathbb{R}} \times \mathbb{R} :$ $x - y \in \mathbb{Z}\}$ ← integers

(a) (4 points) Show that C is an equivalence relation.

Reflexive:

Any number minus itself is 0, which is an integer, thus for any pair (x,x) w/ $x \in \mathbb{R}$, $(x,x) \in C$ making it reflexive

Symmetric:

If $(x,y) \in C$, then $x-y = \text{integer} = -(y-x)$, therefore $y-x$ must also be an integer, so $(y,x) \in C$ and this relation is symmetric.

Transitive:

If $(x,y), (y,z) \in C$, then $x-y = \text{integer}$ and $y-z = \text{integer}$ another. $x-y+(y-z) = \text{integer1} + \text{integer2} = x-z = \text{integer3}$ b/c an integer minus another integer is always an integer. Thus, if $(x,y),(y,z) \in C$, then $(x,z) \in C$ making C transitive.

Since C is Reflexive, Symmetric, and Transitive, it's an equivalence relation.

Question 3 continues on the next page...

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.

Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

Function: for each $x \in X$ there is a unique $y \in Y$ where $f(x) = y$

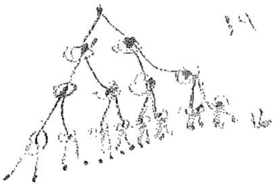
$f(x)$ is injective since $f(a_1) = a_1 + \frac{1}{2}$ and $f(a_2) = a_2 + \frac{1}{2}$ if $f(a_1) = f(a_2) \to a_1 + \frac{1}{2} = a_2 + \frac{1}{2} \to a_1 = a_2$, so for every a there's only one b. since [a] are all numbers that are an integer distance from a and same w/ [b] (all members are integer distance from b) then if $f(a) = b$ then every element in [a] will have a unique corresponding one in [b] satisfying the function. Thus for every [a], where $a \in \mathbb{R}$, it will only map to one [b], making $\tilde{f}$ a function. This is b/c there is a uniform distance between consecutive f(x) values

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

$g(a) = a^2$ would have [a] mapped to multiple [b]'s since the distances between consecutive g(a) change so each value in [a] can associate with a different [b] set due to the nature of the C relation.

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?



$\boxed{i+m \text{ terminal vertices}}$

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \leq m^h$.

Base case: ~~0~~ $h=0$, $t=1$ for any m and $1 \leq m^0 \to 1 \leq 1$ ✓
Base case is true.
Inductive step: Assume $t \leq m^h$, then adding another
layer will cause $t = mt$ since it's a full m-ary tree.
Thus $mt \leq m^{h+1} \Rightarrow m(t) \leq m(m^h) \to t \leq m^h$

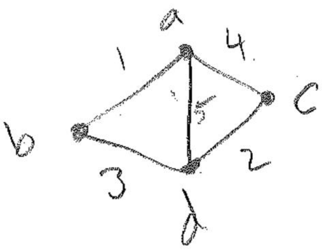
~~Thus if $t \leq m^h$ is true for a given full m-ary~~
Thus if $t \leq m^h$ is true for given
tree of height h, then it's also true for given
full m-ary tree of height h+1. Thus
$t \leq m^h$ is true for any full m-ary tree
of height h and t terminal vertices. □

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

Since G is connected, we know that it has a minimal spanning tree. Let's assume that there are 2 minimal spanning trees for G where all edges have distinct weights. Let's say you find one minimal spanning tree T. You can the connect any two vertices to create a cycle and then remove an original edge in T to create a new spanning tree T' however, because all the weights are distinct, weight(T') ≠ weight(T) meaning that one tree will be "heavier" than the other, which contradicts the claim that there are 2 min spanning trees. The same logic is applicable to having 3+ min spanning trees. Thus by contradiction G must have a single unique min spanning tree if all its edges have distinct weights

Question 5 continues on the next page...

(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.



6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

Every edge can be a part of at most 2 faces.

Every face must be enclosed by ~~made of~~ at least 3 edges

Thus $2E \geq 3F$

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

For any simple planar graph,

$V - E + F = 2 \rightarrow E = V + F - 2$

$\rightarrow 3E = 3V + 3F - 6$

$2E \geq 3F$

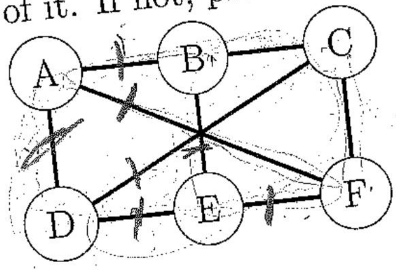
since $2E \geq 3F$ (part (a))

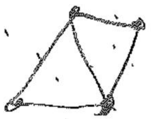
$3E \leq 3V + 2E - 6$

$\boxed{E \leq 3V - 6}$

Question 6 continues on the next page...

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.





V = 4

E = 5

F = 3

Not planar if a subgraph ~~[illegible]~~ can be reduced to K_5 or $K_{3,3}$.

This graph is ~~[illegible]~~ planar b/c it can't be series reduced to K_5 or $K_{3,3}$

~~remove C: B⇔C⇔F B⇔C⇔D F⇔C⇔D~~

↓ ↓ ↓

~~B⇔F B⇔D F⇔D~~

19 = 4F

4 = 2 = 4.5 = 5

Also fulfills Euler's Thm

V − E + F = 2

6 − 9 + 5 = 2

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Base case: $n=1$, $7^1-1=6$ which is divisible by 6, thus base case satisfied.

Inductive step: Assume 7^n-1 is div by 6, then we want to prove $7^{(n+1)}-1$ also div by 6:

$7^{(n+1)}-1 = 7(7^n)-1 = 7(7^n)-7+6$

$= 7(7^n-1)+6 \Rightarrow$ since 7^n-1 div by 6,

$\rightarrow 7^n-1 = 6 \cdot X$ for some integer X

$7(7^n-1)+6 = 7 \cdot 6(X)+6 = 6(7X+1)$ which is div by 6, therefore 7^n-1 is div by 6 for all $n \geq 1$ $\square$

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7.

$\sum_{i=0}^{n} 10^i = 7m$ for some m.

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$

Since the trees for V_{n-1} and V_{n-2} must both be balanced, and they differ by a height of 1, they can simply be chained together with ~~[illegible]~~ a parent root. Thus, you add the vertices of the two older trees plus 1 for the new root node

$V_n = V_{n-1} + V_{n-2} + 1$

Ex:

$V_0 = 1$ $V_1 = 2$ $V_2 = 4 = V_0 + V_1 + 1$ $V_3 = 7 = V_1 + V_2 + 1$

. . .

(b) (3 points) Show that for $n \geq 0$, $v_n = \cancel{F_{n+2}}$ where F_k is the k^{th} Fibonacci number.

$V_n = F_{n+3} - 1$

$V_n = V_{n-1} + V_{n-2} + 1$ → define $Z_n = V_n + 1$

$V_0 = 1,\ V_1 = 2$

$Z_n - 1 = V_n$

then

$Z_n \cancel{-1} = Z_{n-1} \cancel{-1} + Z_{n-2} \cancel{-1}\ \cancel{+1}$

$Z_n = Z_{n-1} + Z_{n-2}$ with $Z_0 = 2, Z_1 = 3$

The Fibonacci sequence's recurrence relation is also

$F_n = F_{n-1} + F_{n-2}$ but $F_0 = 0$ and $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$

Here, $Z_0 = F_3$ and $Z_1 = F_4$, so $Z_n = F_{n+3}$ but since $Z_n = V_n + 1$

$V_n + 1 = F_{n+3}$ → $\boxed{V_n = F_{n+3} - 1}$

Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$V_n = F_{n+3} - 1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+3} - 1$$

$$\leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} \leq \frac{\sqrt{5}+5}{10}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}$$

then $V_n \leq \frac{\sqrt{5}+5}{10}(\varphi^{n+2})$

so $\underline{V_n = O(\varphi^{n+2})}$

$$V_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{n+3} - 1$$

since $\frac{1+\sqrt{5}}{2} > \frac{1-\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2} \geq 1$,

$$V_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{n+3} - 1 \geq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \left(\frac{1+\sqrt{5}}{2}\right)^{n+3}$$

$$V_n \geq \cancel{\frac{1}{\sqrt{5}}(\varphi)^{n+3}} - \cancel{\frac{1}{\sqrt{5}}(\varphi)^{n+3}} - \varphi^{n+3}$$

$V_n \geq -\left(\frac{1+\sqrt{5}}{2}\right)\varphi^{n+2}$ therefore $\underline{V_n = \Omega(\varphi^{n+2})}$

Since $V_n = O(\varphi^{n+2})$ and $V_n = \Omega(\varphi^{n+2})$, $\boxed{V_n = \Theta(\varphi^{n+2})}$

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

$(x+2)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} 2^i$ b/c Binomial thm.

Let $x=1$: $(1+2)^n \boxed{= 3^n = \sum_{i=0}^{n} \binom{n}{i} 2^i}$

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n-1}{r-1}$

$$\sum_{i=0}^{r} \frac{n!\,m!}{(i!)(r-i)!\,(n-i)!\,(m-r+i)!} = n!\,m! \sum_{i=0}^{r} \frac{1}{\cancel{\quad}}$$

$$= n!\,m!\left(\frac{1}{r!\,n!\,(m-r)!} + \frac{1}{(r-1)!\,(n-1)!\,(m-r+1)!} + \cdots\right)$$

$$= \frac{\cancel{n!}\,m!}{r!\,\cancel{n!}\,(m-r)!}\left(1 + (r)(n)(m-r) + (r)(r-1)(n)(n-1)(m-r)(m-r-1) + \dots\right)$$

$$= \binom{m}{r}$$