# Math 61-1 Final exam

DEVIN YERASI

TOTAL POINTS

**67.5 / 90**

QUESTION 1

## Multiple choice 10 pts

1.1 **2 / 2**

✓ **- 0 pts Correct (c)**

**- 2 pts** Incorrect

1.2 **2 / 2**

✓ **- 0 pts Correct (a)**

**- 2 pts** Incorrect

1.3 **2 / 2**

✓ **- 0 pts Correct (c)**

**- 2 pts** Incorrect

1.4 **2 / 2**

✓ **- 0 pts Correct (b)**

**- 2 pts** incorrect

1.5 **2 / 2**

✓ **- 0 pts Correct (a)**

**- 2 pts** Incorrect

QUESTION 2

## Short answer 10 pts

2.1 **2 / 2**

✓ **- 0 pts Correct ((-2)^100 + 3^100)**

**- 1 pts** Almost correct (small arithmetic error in answer)

**- 2 pts** Incorrect

2.2 **0 / 2**

**- 0 pts** Correct (C(7,4)6!4!)

**- 1 pts** Close

✓ **- 2 pts Incorrect**

2.3 **2 / 2**

✓ **- 0 pts Correct (24C4)**

**- 1 pts** Close

**- 2 pts** incorrect

2.4 **0 / 2**

**- 0 pts** Correct

**- 1 pts** Close (Three of four)

✓ **- 2 pts Incorrect**

2.5 **1 / 2**

**- 0 pts** Correct (2^(n^2 - n) + 2^(n^2 +n / 2) - 2^(n^2 - n / 2))

✓ **- 1 pts Close**

**- 2 pts** Incorrect

QUESTION 3

## Equivalence relation 10 pts

3.1 it is an equivalence relation **4 / 4**

✓ **- 0 pts Correct**

**- 1 pts** issue in transitivity

**- 3 pts** misunderstanding of what relation is saying

**- 4 pts** blank

**- 2 pts** misunderstanding of symmetry

**- 1 pts** the decimal thing isn't exactly right, e.g. -.3 is related to .7

**- 0 pts** Click here to replace this description.

**- 1 pts** issue with symmetry

3.2 defining a function **2 / 4**

**- 0 pts** Correct

**- 4 pts** blank

**- 2 pts** need to prove uniqueness part of function

**- 2 pts** missing existence part of function

✓ **- 1 pts issue with uniqueness part of function**

✓ **- 1 pts need to consider different elements in the same equivalence class**

**- 1 pts** thing with decimals isn't quite right, for example -.3 and .7 are related

**- 3 pts** big misunderstanding of the equivalence relation or function

### 3.3 a function that doesn't descend **1 / 2**

**- 0 pts** Correct

**- 2 pts** your g is not a function

✓ **- 1 pts issue with justification**

**- 1 pts** your g does not work

**- 2 pts** blank

QUESTION 4

## m-ary tree 10 pts

### 4.1 number of internal vertices **2 / 5**

**- 0 pts** Correct

✓ **- 1 pts No/incorrect answer**

**- 4 pts** No/incorrect justification

**- 2 pts** Didn't justify number of total vertices

**- 3 pts** "Proof by example"

✓ **- 2 pts Assumed every terminal vertex had the same height as the tree**

**- 5 pts** Nothing

**- 1 pts** Forgot to account for root

**- 2 pts** Didn't subtract off internal vertices

### 4.2 height **3 / 5**

**- 0 pts** Correct

**- 1 pts** No base case

**- 1 pts** Didn't set up/invoke induction

**- 1 pts** Backwards inductive step (didn't show inductive construction is exhaustive)

**- 2 pts** Compared to complete tree without showing this case is extremal

**- 3 pts** Assumed tree is complete / inductive construction forms complete trees from complete trees

**- 1 pts** Assumed all immediate subtrees have height h-1

**- 4 pts** "Proof by example"

**- 5 pts** Nothing shown / Incorrect reasoning

**- 2 Point adjustment**

What is m_1? What is t_h? How did you get the boxed (and incorrect) equation? Inductive step very unclear.

QUESTION 5

## spanning trees 10 pts

### 5.1 unique mst **3 / 6**

**- 0 pts** Correct

**- 3 pts** Appeal to Prim's or Kruskal's Algorithm (without proving it can generate any MST)

**- 6 pts** No / Invalid reasoning

**- 3 Point adjustment**

(-1) Didn't show that your inductive construction is exhaustive

(-0.5) Unclear inductive hypothesis

(-0.5) Assumed your desired conclusion in your inductive step! I don't think you meant to do so, so I'm being generous with this one.

(-1) What if there's an MST of G' involving two different edges incident with v_0?

### 5.2 non unique spanning tree **4 / 4**

✓ **- 0 pts Correct**

**- 4 pts** Not an example

**- 4 pts** Claimed no such graph exists

**- 4 pts** Nothing

QUESTION 6

## planar graphs 10 pts

### 6.1 2e > 3f **3 / 3**

✓ **+ 3 pts Correct**

**+ 2 pts** >= 3 edges for each face

**+ 1 pts** >= 3 edges for each face (w/ mistake)

+ **1 pts** <=2 faces for each edge

+ **0 pts** Incorrect

### 6.2 e<3v-6 **3 / 3**

✓ + **3 pts** Correct

+ **2 pts** Euler's formula

+ **1 pts** Correct application with (a)

+ **0 pts** Incorrect

### 6.3 nonplanar graph **0 / 4**

+ **4 pts** Correct

+ **3 pts** Isomorphic to K_3,3

+ **2 pts** Mistaken/missing ismorphism to K_3,3

+ **1 pts** E <= 2v-4 or 2E >= 4F

+ **1 pts** Other partial credit

✓ + **0 pts Incorrect**

QUESTION 7

10 pts

### 7.1 7^n-1 divisible by 6 **5 / 5**

✓ + **5 pts** Correct

+ **1 pts** Base case

+ **1 pts** Inductive hypothesis

+ **2 pts** factoring out a 7 in inductive step as (6+1) or adding/substracting 7

+ **1 pts** Conclusion

+ **0 pts** Incorrect

### 7.2 number with only 1s divisible by 7 **5 / 5**

✓ + **5 pts** Correct

+ **0 pts** Click here to replace this description.

+ **1 pts** Look at 8 consecutive terms

+ **1 pts** Pigeonhole remainder

+ **1 pts** 7 divides a number of the form 111..000...

+ **2 pts** This implies that 7 divides 10^k*11...

+ **1 pts** Unsuccessful attempt with substantial work

QUESTION 8

## balanced binary trees 10 pts

### 8.1 **4 / 4**

✓ - **0 pts** Correct

- **2 pts** incomplete, need to describe how a height n minimal balanced binary tree is made out of ones of smaller height

- **3 pts** can't just do examples

- **4 pts** blank

- **1 pts** how are you adding in these trees/ vertices?

- **3 pts** can't do induction without using some properties of minimal balanced binary trees

- **4 pts** incorrect numbers/ equation

### 8.2 relationship to fibonacci numbers **2 / 3**

- **0 pts** Correct

- **1.5 pts** that is not the recurrence/ equation for the fibonacci numbers/ minimal balanced binary trees

- **1 pts** you are assuming the desired conclusion

- **3 pts** blank

- **1.5 pts** need to use recurrence for fiboacci numbers

- **1.5 pts** missing inductive step

✓ - **1 pts the two recurrences aren't exactly the same, you need to account for this difference**

- **0.5 pts** error in equations

- **1 pts** need to check initial conditions

### 8.3 Theta **1.5 / 3**

- **0 pts** Correct

✓ - **0.5 pts need to account for other term in equation for fibonacci numbers (sometimes it is contributing something positive, something something negative)**

- **2 pts** wrong formula for fibonacci numbers/ v_n

- **1 pts** issue with big O

✓ - **1 pts issue with omega**

- **3 pts** blank/ no gradable work

- **1 pts** wrong equations/ issues with constants

- **2 pts** need to use equation for v_n/ Fibonacci numbers

QUESTION 9

## binomial coefficients 10 pts

### 9.1 3^n 4 / 4

- ✓ **+ 4 pts** **Correct**
- **+ 3 pts** Minor error
- **+ 2 pts** Binomial theorem
- **+ 1 pts** Attempted induction or counting argument
- **+ 0 pts** Incorret

### 9.2 vandermonde identity 6 / 6

- ✓ **+ 6 pts** **Correct**
- **+ 5 pts** Minor errror
- **+ 3 pts** One part of counting argument or (x+y)^n+m
- **+ 1 pts** Attempted to use induction/binomial thrm/Pascal's identity
- **+ 0 pts** Incorrect

# Final

So back for 1.1, 1.2
de, 36

Name: Devin Yerasi

Student ID: 305167818

Section:

| | Tuesday: | Thursday: | |
|---|---|---|---|
| | 1A | 1B | TA: Albert Zheng |
| | 1C | 1D | TA: Benjamin Spitz |
| | 1E | 1F | TA: Eilon Reisin-Tzur |

**Instructions:** Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.
**Please get out your id and be ready to show it during the exam.**

Please do not write below this line.

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| Total: | 90 | |

$n=3$ $\quad \frac{6}{2} + \frac{6}{2(1)} = 3+3=6$

$k=1$

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   nk

   1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$ $\qquad \binom{n}{k} + \binom{n}{k+1} = \binom{n+k}{k}$

      (a) $\cancel{\frac{(n+k)!}{k!n!}}$

      (b) $\cancel{\frac{(n+1)!}{k!(n+1-k)!}}$

      (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$ $\qquad \frac{(k+1)\,n!}{(k+1)!\,(n-k)} + \frac{n!\,(n-k)}{(k+1)!\,(n-k)!} =$

      (d) none of the above $\qquad \frac{4!}{3!} \quad \frac{4!}{2!\,2!}$

      C or d

2. The decision tree of a sorting algorithm for sorting $n$ items (where at each step we can only decide whether or not one item is less than — other) necessarily has: $\quad n!$ terminal vertices

   (a) a height of $\geq \lg(n!)$

   (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)

   (c) a height of $O(\lg(n!))$ $\qquad h \geq \lg(t)$

   (d) a height of $O(n \lg n)$ $\qquad h \geq \lg(n!)$

   $\Omega(n \lg n)$

3. If $G$ is a graph with $n$ vertices and $n-2$ edges, then:

   (a) $G$ is a tree

   (b) $G$ is connected

   (c) $G$ is disconnected

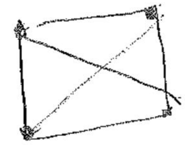
   (d) $G$ is simple

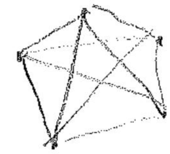
Question 1 continues on the next page...

δ(every vertex) = even

4. Which of these graphs has an Euler ~~cycle~~ every cycle?
   - ~~(a) $K_4$~~
   - (b) $K_5$
   - ~~(c) $K_{3,3}$~~ not planar
   - ~~(d) $K_{2,3}$~~

every vertex has even degree

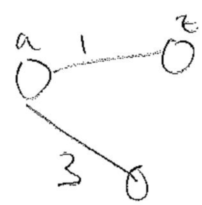




5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with $n$ vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all $n$.

   Best case = d edges ∂(v_i) − 1

   - (a) Less than or equal to $n$
   - (b) More than $n$ but less than or equal to $n^2/2$
   - (c) More than $n^2/2$ but less than or equal to $n^2$
   - (d) More than $n^2$



M_M_M_M_M_M_M_

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

$s_n - s_{n-1} - 6s_{n-2} = 0$ $\quad t^2 - t - 6t = 0$ $\quad (t-3)(t+2)$ $\quad t^2 - 6 - 3t + 2t$

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is $s_{100}$?

$s_n = (3)^n + (-2)^n$ $\qquad s_{100} = (3)^{100} + (-2)^{100}$

$r = 3, -2$
$s_n = a3^n + b(-2)^n$
$2 = a + b$
$1 = 3a - 2b$
$1 = 3(2-b) - 2b$
$1 = 6 - 5b$
$b = 1$
$a = 1$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

7

$7(6!) \times P(8,4)$

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

$\binom{20+5-1}{5-1} = \binom{24}{4}$

$\frac{8!}{4!4!5}$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.

$\frac{C(2n,n)}{n+1} = \frac{C(8,4)}{5} =$

See back of the page

(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with $n$-elements?

$2^{n^2}$ total relations, $2^{n^2-n}$ reflexive, $2^{\frac{n^2}{2}}$ symmetric, $2^{\frac{n^2-n}{2}}$ symm + reflex

$2^{\frac{n^2}{2}} + 2^{n^2-n} - 2^{\frac{n^2-n}{2}}$

$n+n$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$.

$y = x + C_1$

$z = C_2 + y$

(a) (4 points) Show that $C$ is an equivalence relation.

Symmetric:

- let $(x, y) \in C$, $x - y \in \mathbb{Z}$ and $(x, y) \in \mathbb{R} \times \mathbb{R}$
- since $(x, y) \in \mathbb{R} \times \mathbb{R}$, $x, y \in \mathbb{R}$, so $(y, x) \in \mathbb{R} \times \mathbb{R}$
- since $x - y \in \mathbb{Z}$, $y - x \in \mathbb{Z}$ as $y - x$ is simply $-(x - y)$ and negative and positive [illegible] exist in $\mathbb{Z}$

therefore if $(x, y) \in C$, $(y, x) \in C$ so $C$ is symmetric

transative:

- along with $(x, y) \in C$, let $(y, z) \in C$ in the same way
- since $x - y \in \mathbb{Z}$ there is a constant $C_1 \in \mathbb{Z}$, the $x - y = C_1$ and a constant $C_2 \in \mathbb{Z}$, $y - z = C_2$

therefore $x = C_1 + y$ and $z = y - C_2$

- consider the [illegible] $(x, z)$, $(x, z) \in C$ if $(x - z) \in \mathbb{Z}$

and $((x) - (z)) = ((C_1 + y) - (y - C_2)) = C_1 + C_2$

since $C_1$ and $C_2 \in \mathbb{Z}$ their sum must also $\in \mathbb{Z}$,

therefor $(x - z) \in \mathbb{Z}$, and since $z \in \mathbb{R}$,

$(x, z) \in C$ so $C$ is transative

reflexive:

let $(x, y) \in C$. Since $C$ has been proven to be symmetric $(y, x) \in C$ and since $C$ was proven to be transative $(x, x) \in C$, so $C$ is reflexive

Since $C$ is symmetric, transative, and reflexive, it is an equivalence relation

Question 3 continues on the next page. . .

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of $C$, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.

Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

as $[a]$ and $[b]$ are equivalance classes they are disjoint

- as $f(x) = x + 1/2$, it maps each unique $x$ to a value is greater than the input, which when done to a set of unique values results in another set of unique values different than the previous

therefore no two equivalence classes $[a_1], [a_2]$ can map to the same resulting equivalence class $[b]$, as they must all be disjoint, so ~~each of them~~ $[a_1] \cap [a_2] = \emptyset$ if $[a_1] \neq [a_2]$

so $\tilde{f}$ is a function

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

$(-1)^{-1} = \frac{1}{(-1)^1} = -1$

$g(x) = 2x - x^2$

~~$g(2) = 2(2) - 2^2 = 0$~~

$g(0) = 2(0) - (0)^2 = 0$

two numbers $\in \mathbb{R}$ map to same $b$, so $g$ is not a function so $\tilde{g}$ is not a function

$h_{n+1} = i$

4. For $m$ a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly $m$ children.

   (a) (5 points) If $T$ is a full $m$-ary tree with $i$ internal vertices, how many terminal vertices does $T$ have?

$m^h = i - i_{n-1}$

| h | i | terminal vers. |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | m |
| 2 | m+1 | $m^2$ |
| 3 | $m^2+m+1$ | $m^3$ |
| 4 | $(m^3+m^2+m+1)$ | $m^4$ |

$i = \sum_{k=1}^{h-1} m^k$

terminal vertices $= m^h$, $h$ = height of tree

$t = \left(\sum_{k=0}^{h} m^k\right) - i = m^h$

m a binary tree

| h | i | t |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 2 |
| 2 | 3 | 4 |

$t = i + 2$

$v = 2i + 2 = i + i + 2$

   (b) (5 points) Show that if $T$ is a full $m$-ary tree of height $h$ with $t$ terminal vertices, then $t \leq m^h$.

Proof by induction:

Base Case: height $= 0$, so 1 terminal vertex (the root)

$1 \leq m^0 = 1$ ✓

Induction: Assume true for a height $k$, $0 \leq k < h$ so that at most $= h-1$

a full m-ary tree (G) of height $h$'s terminal vertices can be thought of as the sum of the $m$ sub-m-ary trees's starting with the children of the root of G, $\boxed{t_h = t_{h-1} + t_{h-1} + \cdots + t_{h-1}}$

so $t_h \leq m(t_{h-1})$

since $t_{h-1} \leq m^{h-1}$

$t_h \leq m(m^{h-1}) = m^h$

So

$t_h \leq m^h$ for all $h \geq 0$

5. (a) (6 points) Show that if $G$ is a connected weighted graph where all the edges of $G$ have distinct weights then $G$ has a unique minimal spanning tree.

Proof by induction:

Base case: if vertices = 1, the unique spanning tree is simply the single vertex so the argument holds true

Induction: Assume true for all k vertices $1 \leq k \leq n$

Consider G, a connected weighted graph with n+1 vertices and a unique minimal spanning tree. where to be added to it from a single vertex (Vo) to G by forming G', along with an arbitrary # of edges having Vo to other vertices such that all the edges added were distinct from each other and those already in G, a unique minimal spanning tree of G' can be formed by first considering the unique minimal spanning tree of G, lets call it $T_G$ and the unique minimal spanning tree of G', $T_{G'}$.

By adding the edge with a minimum weight linking Vo to a vertex on $T_G$ and excluding all other edges going to form G', we form a unique minimal spanning tree for G', called $T_{G'}$. As each edge is distinct, there is ~~only~~ only one possible choice of what vertex in G (and $T_G$) to link to Vo to maintain a minimal spanning tree, making $T_{G'}$ unique provided $T_G$ is unique.

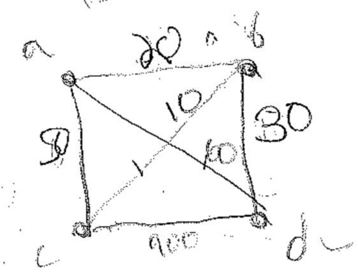
Therefore if G is a connected weighted graph with all edges having distinct weights, then G has a unique minimal spanning tree.

Question 5 continues on the next page...

(b) (4 points) Give an example of a connected weighted graph $G$ so that all the edges of $G$ have distinct weights and $G$ has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.

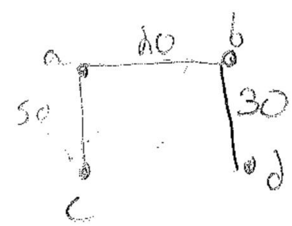
~~Prove no such Graph exists~~

$\underline{G}$

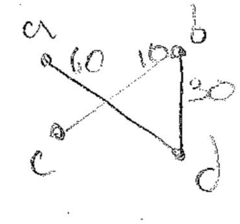



70
30, 40
20, 50
10, 60

Spanning tree 1 w = 100




Spanning tree 2 w = 100



6. (a) (3 points) Show that for $G$ a connected simple planar graph containing a cycle if $G$ has $E$ edges and $F$ faces, then $2E \geq 3F$.

$F = e - v + 2$

each cycle must be bound by at least 3 edges, otherwise a cycle cannot form in a simple graph without loops or parallel edges. Each edge bounding a face (cycle) can at most bound 2 faces in a planar graph

So $2E \geq 3F$

(b) (3 points) Show that for $G$ a connected simple planar graph containing a cycle if $G$ has $E$ edges and $V$ vertices, then $E \leq 3V - 6$.

$F = E - V + 2$

using the above expression proved in part a $(2E \geq 3F)$

$2E \geq 3F = 3(E - V + 2)$

$2E \geq 3E - 3V + 6$

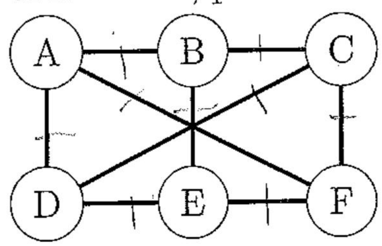
$3V - 6 \geq E$

$\Updownarrow$

$E \leq 3V - 6$

Question 6 continues on the next page. . .

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.




$V = 5$

$E = 9$

Proof By contradiction:

Assume the graph above, (G) is planar. As it has 5 vertices and 9 edges:

$F = E - V + 2 = 9 - 5 + 2 = 7$ faces

it must have 7 faces, provided it is planar,

Since the graph above is simple (no parallel edges + loops), and is planar, it must follow the inequality proven in part a ($2E \geq 3F$)

$2E \geq 3F$

$2(9) \geq 3(7)$

$18 \geq 21$ is not correct, providing a contradiction

Therefore the graph above cannot be planar

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Proof by Induction;

Base Case:

$n=1$ $\quad 7^1 - 1 = 6$ , 6 divisible by 6

Induction: assume $7^n - 1$ is divisible by 6 for all $k$, $1 \leq k \leq n$

$7^{n+1} - 1 = 7(7^n) - 1$

Since $7^n - 1$ is divisible by 6, $7^n - 1 = 6k$, such that $k \in \mathbb{N}$

so $7^n = 6k + 1$ for some arbitrary $k$

$7(7^n) - 1 = 7(6k+1) - 1 =$

$7(6k) + 6$

since each term is divisible by 6, the sum must be divisible by 6 so $7^{n+1} - 1$ is divisible by 6 , so $7^n - 1 \mod 6 = 0$, $n \geq 1$

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7.

$$
\begin{array}{r}
15873 \\
7\overline{)111111} \\
-7 \\
\hline
41 \\
-35 \\
\hline
61 \\
56 \\
\hline
51 \\
49 \\
\hline
21
\end{array}
$$

$111111 = \sum_{i=0}^{5} 10^i$

and $\frac{111111}{7} = 15873$

$$
\begin{array}{r}
15873 \\
7\overline{)111111} \\
7 \\
\hline
41 \\
35 \\
\hline
61 \\
56 \\
\hline
51 \\
49 \\
\hline
21
\end{array}
$$

I really did not expect that to work

alternatively, one could argue that if one has only a arbitrarily long number of 1's 6, 7, the remainder at each step would alternate between numbers and eventually reach 2, so by doing the next 1 division one would get 21 which is divisible by 7.

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let $v_n$ denote the minimum number of vertices in a balanced binary tree of height $n$.

(a) (4 points) Show that $v_n$ satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$    $V_n = V_{n-1} + V_{n-2} + 1$

Induction: $V_0 = 1$, $V_1 = 2$
Base case: $n = 2$    $V_2 = V_1 + V_0 = 1 + 2 + 1 = 4$


✓.

Induction: Assume true for all $k$, $2 \leq k \leq n$
example: the case of $h = n+1$ in Graph $G$. Two subgraphs of $G$, called $G_1$ and $G_2$, can be formed by $G$ by simply removing the root and having each child of the root being a new root to its own tree. In order for these to be a minimal number of vertices, in $G$, one of the subgraphs $G_1$, $G_2$ must have a smaller height than the other. Lets call $G_1$'s height $n$ and $G_2$'s height $n-1$. Since each of these heights are within the assumed bounds, we can use the recurrence formula to find the min vertices of each called $V_n$ and $V_{n-1}$. So, the minimum # of vertices in $G$ is simply these two added plus the root node, so $g_{n+1} = V_n + V_{n-1} + 1$

when proves $V_n = V_{n-1} + V_{n-2} + 1$ for all $n \geq 2$

(b) (3 points) Show that for $n \geq 0$, $v_n = \cancel{F_{n+2}}$ $F_{n+3} - 1$, where $F_k$ is the $k^{th}$ Fibonacci number.

$V_0 = 1$  ⇒ $F_{0+3} - 1 = 1$
$V_1 = 2$  ⇒ $F_{1+3} - 1 = 2$

and $V_n$ follows the same recurrence as the Fibonacci sequence with $V_n = V_{n-1} + V_{n-2}$ (+1), the (+1) is canceled by the (+1) of $V_n = F_{n+3} - 1$, so $V_n = F_{n+3} - 1$, $n \geq 0$

Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

$V_n = F_{n+3} - 1 = V_{n-1} + V_{n-2} + 1$

it $V_n = V_{n-1} + V_{n-2} + 1 \geq V_{n-1} + V_{n-2}$

$t^2 - t - 1 = 0$

$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

so $V_n \geq a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$

since $V_n = O(\phi^{n+2})$ and $V_n = \Omega(\phi^{n+2})$, it follows that $V_n = \Theta(\phi^{n+2})$

Partial Credit?

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

Binomial Theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

take $a = 1$, $b = 2$

$$(1+2)^n = (3)^n = \sum_{i=0}^{n} \binom{n}{i} (1)^{n-i} (2)^i = \sum_{i=0}^{n} 2^i \binom{n}{i}$$

so $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

RH: # of ways to chose r items from a set of n+m items

LH: The $\binom{n}{i}\binom{m}{r-i}$ inside the sigma details the # of ways to chose i items first from the first n items in the set of size n+m × # of ways to choose the remaining r−i items from the remaining m items in th n+m size set, giving the total # of combinations of the first i chosen items in the first set subset and the remaining items from the second subset of size m. The summation adds every way of chosing a certain number from the first set and the remaining from the second set, resulting in the number of ways to chose r items from a set of n+m items, so the right hand side (RH) = the left hand side (LH).

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