Final

Name: [redacted]

Student ID: [redacted]

Section:

Tuesday:	Thursday:	
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1C	1D	TA: Benjamin Spitz
1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

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Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$
      (a) $\frac{(n+k)!}{k!n!}$
      (b) $\frac{(n+1)!}{k!(n+1-k)!}$
      (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$
      (d) none of the above

      $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}$

      $\frac{n!(k+1) + n!(n-k)}{(k+1)!(n-k)!} = \frac{n!(k+1+n-k)}{(n-k)!(k+1)!}$

      $= \frac{n!(n+1)}{(n-k)!(k+1)!} = \frac{(n+1)!}{(k+1)!(n-k}$

   2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:
      (a) a height of $\geq \lg(n!)$
      (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)
      (c) ~~a height of $O(\lg(n!))$~~
      (d) ~~a height of $O(n \lg n)$~~

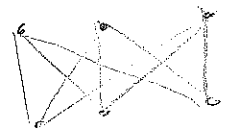
      At most $n!$ options so $\lg(n!)$ minimum

   3. If G is a graph with n vertices and $n-2$ edges, then:
      (a) G is a tree
      (b) G is connected
      (c) G is disconnected
      (d) G is simple

      V.

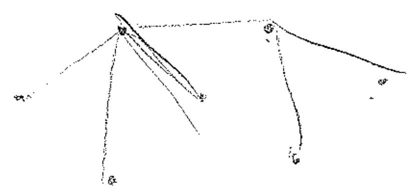
Question 1 continues on the next page...

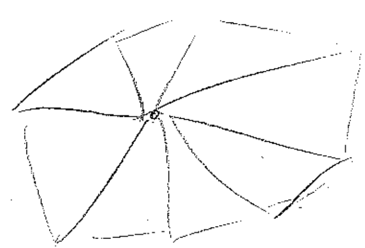
4. Which of these graphs has an Euler cycle?
   (a) K_4
   (b) K_5
   (c) $K_{3,3}$
   (d) $K_{2,3}$

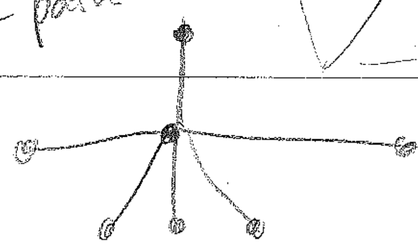


5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.
   (a) Less than or equal to n
   (b) More than n but less than or equal to $n^2/2$
   (c) More than $n^2/2$ but less than or equal to n^2
   (d) More than n^2

A tree, there is a unique simple path between every 2 vertices so theres only one choice for the path







has n-1 ~~comparisons~~ examinations

$c_1 = 2 - c_2$
$1 = 3(2-c_2) - 2c_2$
$1 = 6 - 5c_2$
$c_2 = 1 \quad c_1 = 1$

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

$s_2 = 8$ $\quad 9+4$

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}?

$s_n - s_{n-1} - 6_{n-2} = 0$
$\lambda^2 - \lambda - 6 = 0$
$(\lambda - 3)(\lambda + 2) = 0$
$\lambda = 3$
$\lambda = -2$
$s_n = c_1 3^n + c_2(-2)^n$
$s_0 = 2 = c_1 + c_2$
$s_1 = 1 = 3c_1 - 2c_2$
$s_n = 3^n + (-2)^n$
$\boxed{s_{100} = 3^{100} + (-2)^{100}}$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

1234
$=$
4123
$=$
3412
$=$
2341

$7! \binom{7}{4} - 10$

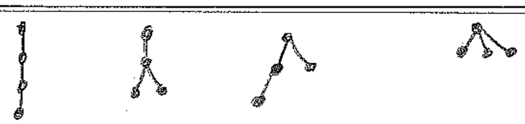
total

number of rotational similarities

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

5^{20}

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.



(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

$2^{n(n-1)} + 2^{\frac{n(n-1)}{2}} - 2^{\frac{(n-1)(n-2)}{2}}$

$A \cup B = A + B - A \cap B$

reflexive relations

symmetric relations

reflexive & symmetric relations

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R}\times\mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

Symmetric

$(x,y) \in \mathbb{R}\times\mathbb{R} : x-y \in \mathbb{Z}$

This means $x-y$ is an integer.

If $x-y$ is an integer, so is $y-x$

because $y-x = -(x-y)$

So therefore yRx by the relation C!

Transitive

~~the~~ If $x-y$ is an integer xRy by C

If $y-z$ is an integer yRz by C

$x-y=n_1$ $y-z=n_2$

$x=n_1+y$ $z=y-n_2$

$x-z = n_1+y-y+n_2$

$= n_1+n_2$

The sum of two integers is also an integer!

This means if xRy by C and yRz by C then xRz by C!

Reflexive

xRx by C if and only if

$x-x \in \mathbb{Z}$.

~~x~~ $x-x=0$ which exists in $\mathbb{Z}$ so therefor xRx by C

Because its transitive, symmetric & reflexive it is an equivalence relation

Question 3 continues on the next page...

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.
Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

For it to be a function, every input needs a single output. $\tilde{f}\ \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a)=b\}$
This means for each $[a]$ we plug in we only 1 $[b]$ that satisfies the condition,
We know that the equivalence classes of C are all numbers whose differences are integers. This means that the equivalence class will be all numbers separated by integers, ie $\{[1.5] = \{1.5, 2.5, 3.5, 4.5 \ldots\}$
for some $[a]$ if we say $f(a)=b$ then you will get a distinct $[b]$ which is $[a+\frac{1}{2}]$. This equivalence class is the only possible output for $[a]$ so therefor it is a function.
For every $[a]$ there is only 1 $[b]$.

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

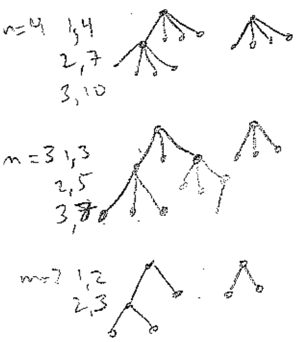
For it not to be a function $g(x) = b$ would have to map to different equivalence classes.

$$g(x) = \begin{cases} 1.5 & x=1 \\ 2 & x \neq 1 \end{cases}$$

so for $a=[1]=[2]$ you could get two different equivalence classes as output, $[1.5]$ or $[2]$!
This is no longer a function because $[1]=[2]$ so they are the same input but produce different output.
$[1] \to [1.5]$ but $[2] \to [2]$!

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?



If it has i internal vertices it has

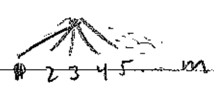
$i(m-1)+1$, terminal vertices

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \le m^h$.

$t = i(m-1)+1$

$\frac{t-1}{m-1} = i$

Claim: A full m-ary tree of height h and t terminal vertices has at most m^h terminal vertices. If we start with a full m-ary tree of height 0, we only have 1 terminal vertex. Height 1 has m terminal vertices.



. If now we consider a tree of height 2, it consists of a height 1 m-ary tree where each child is our height 1 tree from the previous step. For height 3 we have a height 1 tree where each terminal vertex is a height 2 tree. We can do this until h to get a height h tree. If we follow this process, at each height $a \le h$, the number of terminal vertices is m^a so the maximum terminal vertices for h height is m^h. $t \le m^h$

This is $m \cdot m$ total terminal vertices for height 2

This is $m \cdot m \cdot m$ total terminal vertices for height 3

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

Using Prim's algorithm we can show that you will find a unique minimal spanning tree.

If we arbitrarily pick one vertex of the tree to be our starting point, we can begin Prim's algorithm. We add this vertex to our min tree and now look for the edge connected to our vertex with smallest weight to find our next vertex (v_2). Because they all have distinct weights, there is no case such that we have to choose between 2 edges of equal weight. Now we look for ~~vert~~ edges connected to v_1 or v_2 to add, given the fact that we have distinct weights there will be a unique minimal edge. We can keep doing this ~~for~~ $v_3 \cdots v_n$ where we compare adjacent edges to our minimal spanning tree and add the one that connects an undiscovered node with minimal weight. Because of the uniqueness of weights, we will reach the end of this process without having chosen between 2 paths of equal weight. Since we made no decisions, there is only 1 minimal spanning tree.

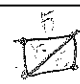
Question 5 continues on the next page...

(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.

No such weighted graph exists, because there is no such graph where all the edges have unique weights and 2 distinct spanning trees exist. We proved this in the previous part by showing that by using prim's algorithm we never have to chouse between 2 edges of the same weight so only 1 minimal spanning tree may exist.

6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

$V - E + F = 2$



Each edge bounds in at least 2 faces. If we have a simple graph the minimum vertices in a cycle is 3 so each face consists of at least 3 edges. We cannot have a case in which  there is a cycle between 2 points because that's not simple.

Therefor

$2E \geq 3F$

Because each edge can border two faces (so we don't count them twice) and we have at least 3F edges because we need at least 3 edges per face.

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

$$2E \geq 3F$$

$$F = 2 + E - V$$

$$2E \geq 3(2 + E - V)$$

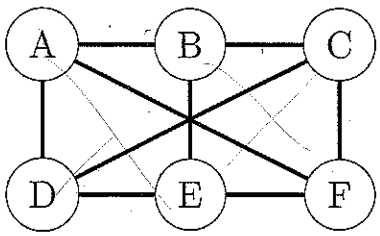
$$2E \geq 6 + 3E - 3V$$

$$-E \geq 6 - 3V$$

$$\boxed{E \leq 3V - 6}$$

Question 6 continues on the next page...

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.



$V=6 \quad E=9$

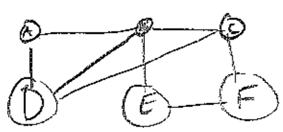
~~$2E \geq 3l$~~

$E \leq 3V-6$

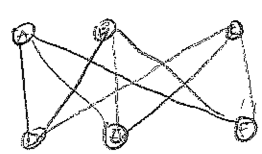
$9 \leq 3(6)-6$

$9 \leq 12$

There is a path from D→B through E so we can draw a new path that represents D→E→B



we could do the same thing for A→B→E
C→B→~~E~~ B
F→~~E~~→B



This related graph is $K_{3,3}$! This means that our graph is not planar as it contains a graph that upon some basic transformations is $K_{3,3}$.

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Base case: $7^1 - 1 = 6$ ✓
6 is divisible by 6
Inductive step: Assume $7^n - 1$ is divisible by 6

$$7^{n+1} - 1 = 7 \cdot 7^n - 1$$
$$= 7 \cdot 7^n - 7 + 6$$
$$= 7 \cdot (7^n - 1) + 6$$

We know $7^n - 1$ is divisible by 6 and if we multiply a number ~~x~~ divisible by 6 by 7 and add 6 we still have a number divisible by 6 for any $n \geq 1$. This means we are done by induction!

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7.

Base 7

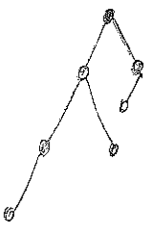
- Needs to end in 0
- 111...11 is of what form?

Base 2

- how to shift?

If we consider a number of all 1's ie 111....11 and we multiply it by 10 and add 1 we get the next all 1 number. Each consecutive number will have a different remainder when divided by 7 because the operation of $10x+1$ does not preserve remainders. If we keep on doing this we are bound to find a number that is divisible by 7 because there are only 7 options for remainders and infinite options for 111....11 ~~x~~ numbers of this form. By pigeonhole principle there is some number that satisfies this condition. $15873 \cdot 7 = 111,111$
This is an example of 1 such number.
This is a pigeonhole principle problem

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.



(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$

$V_n = V_{n-1} + V_{n-2} + 1$

The loger tree we can put on the left and have it be of height h-1. The shorter tree ~~a~~ we can put on the right and have it be of height h-2. The difference in heights is 1. We can recursively do the same thing for the subtrees. This explains V_{n-1} & V_{n-2}. We have to add 1 to include our new root node.

← represents new edge

So for any tree we can say the min balanced tree is $V_n = V_{n-1} + V_{n-2} + 1$

(b) (3 points) Show that for $n \geq 0$, $v_n = F_{n+3} - 1$ where F_k is the k^{th} Fibonacci number.

$V_n = F_{n+3} - 1$

Base case: $V_0 = F_3 - 1$

$= 2 - 1$

$= 1$ ✓

Inductive step:

Assume $V_n = F_{n+3} - 1$

$V_{n+1} = V_n + V_{n-2} + 1$

$= F_{n+3} - 1 + F_{n+2} - 1 + 1 = F_{n+3} + F_{n+2} - 1 = F_{(n+1)+3} - 1$

We are done by strong induction

Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

$V_n = F_{n+3} - 1$

$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$

$F_{n+3} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}$

$V_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2} - 1$

$V_n \leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}$

$C_1 = \frac{1}{\sqrt{5}} \qquad V_n = O\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}\right)$

$V_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2} - 1$

$V_n \geq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}$

$\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} \geq \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}$

$V_n \geq \frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}$

$C_2 = \frac{2}{\sqrt{5}} \quad V_n = \Omega\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}\right)$

$\boxed{O(\phi^{n+2})\ \&\ \Omega(\phi^{n+2}) \text{ means } \Theta(\phi^{n+2})}$

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

We could consider the counting problem of putting n items into 3 groups and looking for all the possible groups. We could either consider each item to have 3 options and randomly assign them (3^n) or we could consider a case in which we choose i of the n items go into the 2nd & 3rd group and we put the rest in the first group. Out of the i we have chosen to go into groups 2 & 3 we can randomly assign each to a group, giving 2^i possibilities. If we sum this among all $i \leq n$ then we will get the total ways to create 3 groups from n items. Since these solve the same counting problem, they must be equal.

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

This makes sense if we consider a counting problem where we choose r distinct things from two groups, n & m, each having at least r items. We could either look at it as choosing r things out of the supergroup n+m which would give $\binom{n+m}{r}$ options. Or we could sum up the total options for choosing i things from n and the remaining r-i things from m. This would give $\sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$ Because these count the same thing, they must be equal!

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