Math 61-1 Final exam

YICHEN LYU

TOTAL POINTS

68 / 90

QUESTION 1

Multiple choice 10 pts

1.1 **2 / 2**

✓ **- 0 pts Correct (c)**

- 2 pts Incorrect

1.2 **0 / 2**

- 0 pts Correct (a)

✓ **- 2 pts Incorrect**

1.3 **2 / 2**

✓ **- 0 pts Correct (c)**

- 2 pts Incorrect

1.4 **2 / 2**

✓ **- 0 pts Correct (b)**

- 2 pts incorrect

1.5 **0 / 2**

- 0 pts Correct (a)

✓ **- 2 pts Incorrect**

QUESTION 2

Short answer 10 pts

2.1 **2 / 2**

✓ **- 0 pts Correct ((-2)^100 + 3^100)**

- 1 pts Almost correct (small arithmetic error in answer)

- 2 pts Incorrect

2.2 **0 / 2**

- 0 pts Correct (C(7,4)6!4!)

- 1 pts Close

✓ **- 2 pts Incorrect**

2.3 **2 / 2**

✓ **- 0 pts Correct (24C4)**

- 1 pts Close

- 2 pts incorrect

2.4 **2 / 2**

✓ **- 0 pts Correct**

- 1 pts Close (Three of four)

- 2 pts Incorrect

2.5 **2 / 2**

✓ **- 0 pts Correct (2^(n^2 - n) + 2^(n^2 +n / 2) - 2^(n^2 - n / 2))**

- 1 pts Close

- 2 pts Incorrect

QUESTION 3

Equivalence relation 10 pts

3.1 it is an equivalence relation **4 / 4**

✓ **- 0 pts Correct**

- 1 pts issue in transitivity

- 3 pts misunderstanding of what relation is saying

- 4 pts blank

- 2 pts misunderstanding of symmetry

- 1 pts the decimal thing isn't exactly right, e.g. -.3 is related to .7

- 0 pts Click here to replace this description.

- 1 pts issue with symmetry

3.2 defining a function **3 / 4**

- 0 pts Correct

- 4 pts blank

- 2 pts need to prove uniqueness part of function

- 2 pts missing existence part of function

- 1 pts issue with uniqueness part of function

✓ - 1 pts **need to consider different elements in the same equivalence class**

- **1 pts** thing with decimals isn't quite right, for example -.3 and .7 are related

- **3 pts** big misunderstanding of the equivalence relation or function

3.3 a function that doesn't descend 1 / 2

- **0 pts** Correct

✓ - 2 pts **your g is not a function**

- **1 pts** issue with justification

- **1 pts** your g does not work

- **2 pts** blank

+ 1 **Point adjustment**

what about 0?

QUESTION 4

m-ary tree 10 pts

4.1 number of internal vertices 1 / 5

- **0 pts** Correct

✓ - 1 pts **No/incorrect answer**

- **4 pts** No/incorrect justification

- **2 pts** Didn't justify number of total vertices

- **3 pts** "Proof by example"

✓ - 2 pts **Assumed every terminal vertex had the same height as the tree**

- **5 pts** Nothing

- **1 pts** Forgot to account for root

- **2 pts** Didn't subtract off internal vertices

- 1 **Point adjustment**

Argument unclear

4.2 height 2 / 5

- **0 pts** Correct

- **1 pts** No base case

✓ - 1 pts **Didn't set up/invoke induction**

- **1 pts** Backwards inductive step (didn't show inductive construction is exhaustive)

✓ - 2 pts **Compared to complete tree without showing this case is extremal**

- **3 pts** Assumed tree is complete / inductive construction forms complete trees from complete trees

- **1 pts** Assumed all immediate subtrees have height h-1

- **4 pts** "Proof by example"

- **5 pts** Nothing shown / Incorrect reasoning

QUESTION 5

spanning trees 10 pts

5.1 unique mst 2 / 6

- **0 pts** Correct

- **3 pts** Appeal to Prim's or Kruskal's Algorithm (without proving it can generate any MST)

- **6 pts** No / Invalid reasoning

- 4 **Point adjustment**

You don't want to remove the small edges! It's also unclear exactly the process you want to use: where do you start? When do you stop? And how do you know the graph you end with is an MST of the original graph?

5.2 non unique spanning tree 4 / 4

✓ - 0 pts **Correct**

- **4 pts** Not an example

- **4 pts** Claimed no such graph exists

- **4 pts** Nothing

QUESTION 6

planar graphs 10 pts

6.1 2e > 3f 3 / 3

✓ + 3 pts **Correct**

+ **2 pts** >= 3 edges for each face

+ **1 pts** >= 3 edges for each face (w/ mistake)

+ **1 pts** <=2 faces for each edge

+ **0 pts** Incorrect

6.2 e<3v-6 3 / 3

✓ + 3 pts **Correct**

+ **2 pts** Euler's formula

+ **1 pts** Correct application with (a)

+ **0 pts** Incorrect

6.3 nonplanar graph **3 / 4**

+ **4 pts** Correct

✓ + **3 pts** **Isomorphic to K_3,3**

+ **2 pts** Mistaken/missing ismorphism to K_3,3

+ **1 pts** E <= 2v-4 or 2E >= 4F

+ **1 pts** Other partial credit

+ **0 pts** Incorrect

QUESTION 7

10 pts

7.1 7^n-1 divisible by 6 **5 / 5**

✓ + **5 pts** **Correct**

+ **1 pts** Base case

+ **1 pts** Inductive hypothesis

+ **2 pts** factoring out a 7 in inductive step as (6+1) or adding/substracting 7

+ **1 pts** Conclusion

+ **0 pts** Incorrect

7.2 number with only 1s divisible by 7 **5 / 5**

✓ + **5 pts** **Correct**

+ **0 pts** Click here to replace this description.

+ **1 pts** Look at 8 consecutive terms

+ **1 pts** Pigeonhole remainder

+ **1 pts** 7 divides a number of the form 111..000...

+ **2 pts** This implies that 7 divides 10^k*11...

+ **1 pts** Unsuccessful attempt with substantial work

QUESTION 8

balanced binary trees 10 pts

8.1 **4 / 4**

✓ - **0 pts** **Correct**

- **2 pts** incomplete, need to describe how a height n minimal balanced binary tree is made out of ones of smaller height

- **3 pts** can't just do examples

- **4 pts** blank

- **1 pts** how are you adding in these trees/ vertices?

- **3 pts** can't do induction without using some properties of minimal balanced binary trees

- **4 pts** incorrect numbers/ equation

8.2 relationship to fibonacci numbers **3 / 3**

✓ - **0 pts** **Correct**

- **1.5 pts** that is not the recurrence/ equation for the fibonacci numbers/ minimal balanced binary trees

- **1 pts** you are assuming the desired conclusion

- **3 pts** blank

- **1.5 pts** need to use recurrence for fiboacci numbers

- **1.5 pts** missing inductive step

- **1 pts** the two recurrences aren't exactly the same, you need to account for this difference

- **0.5 pts** error in equations

- **1 pts** need to check initial conditions

8.3 Theta **1 / 3**

- **0 pts** Correct

- **0.5 pts** need to account for other term in equation for fibonacci numbers (sometimes it is contributing something positive, something something negative)

- **2 pts** wrong formula for fibonacci numbers/ v_n

✓ - **1 pts** **issue with big O**

✓ - **1 pts** **issue with omega**

- **3 pts** blank/ no gradable work

- **1 pts** wrong equations/ issues with constants

- **2 pts** need to use equation for v_n/ Fibonacci numbers

QUESTION 9

binomial coefficients 10 pts

9.1 3^n **4 / 4**

✓ + **4 pts** **Correct**

+ **3 pts** Minor error

+ **2 pts** Binomial theorem

+ **1 pts** Attempted induction or counting argument

+ **0 pts** Incorret

9.2 vandermonde identity 6 / 6

- ✓ **+ 6 pts** **Correct**
- **+ 5 pts** Minor errror
- **+ 3 pts** One part of counting argument or (x+y)^n+m
- **+ 1 pts** Attempted to use induction/binomial thrm/Pascal's identity
- **+ 0 pts** Incorrect

Final

Name: Yichen Lyu

Student ID: 004940413

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

1. (10 points) Circle the correct answer (only one answer is correct for each question) $\binom{n}{k} + \binom{n}{k+1} \to \binom{n+1}{k+1}$ $\frac{(n+1)!}{(k+1)!(n-k)}$

   1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$
      (a) $\frac{(n+k)!}{k!n!}$
      (b) $\frac{(n+1)!}{k!(n+1-k)!}$
      (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$ $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$
      (d) none of the above

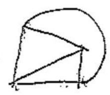
   2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has: $\lg(n!)$ $\lg 3$ $\lg 2$ $\lg 7$
      (a) a height of $\geq \lg(n!)$
      (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)
      (c) a height of $O(\lg(n!))$
      (d) a height of $O(n \lg n)$

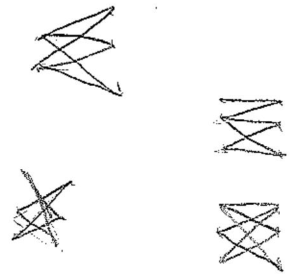
   3. If G is a graph with n vertices and $n-2$ edges, then:
      (a) G is a tree
      (b) G is connected
      (c) G is disconnected
      (d) G is simple

Question 1 continues on the next page...

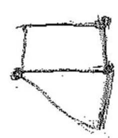
Question 1 continued...

4. Which of these graphs has an Euler cycle?
   (a) K_4
   (b) K_5
   (c) $K_{3,3}$
   (d) $K_{2,3}$





5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.
   (a) Less than or equal to n
   (b) More than n but less than or equal to $n^2/2$
   (c) More than $n^2/2$ but less than or equal to n^2
   (d) More than n^2



n+

13 + 78

$S_3 = S_2$ 13+6=19

$S_4 =$ 19+78=97.

$2^4 = 16$

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

$r^2 - 1r - 6 = (r-3)(r+2)$ $r=-2 / r=3$

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}?

$S_2 = 1 + 12 = 13$ $b(-2)^n + a3^n$

$S_n = (-2)^n + 3^n$ $S_{100} = (-2)^{100} + 3^{100} = 2^{100} + 3^{100}$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

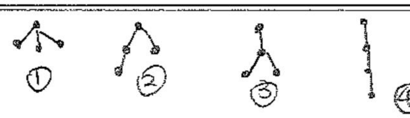
11

$$\frac{7!\binom{8}{4}}{11}$$

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

$C(20+4, 4) = C(24, 4)$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.



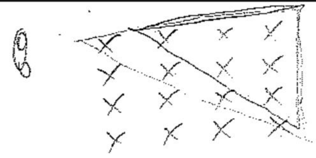
(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

symmetric : $2^{\frac{n^2+n}{2}}$

reflexive : 2^{n^2-n}

both : $2^{\frac{n^2-n}{2}}$

$$2^{\frac{n^2+n}{2}} + 2^{n^2-n} - 2^{\frac{n^2-n}{2}}$$



symmetric : $2^{\frac{n^2+n}{2}}$

reflexive : 2^{n^2-n}

symmetric & reflexive $2^{\frac{n^2-n}{2}}$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R}\times\mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

To prove : C is reflexive :

for $(x,x), \forall x \in \mathbb{R}, x-x=0 \in \mathbb{Z}$.

Thus, C is reflexive

To prove C is transitive

for (x,y) and (y,z), $x-y=a, a\in\mathbb{Z}$

$y-z=b, b\in\mathbb{Z}$

$x-z = x-y+y-z = a+b \in \mathbb{Z}$

Thus, (x,z) exists and C is transitive

To prove C is symmetric

for (x,y) in C, $x-y=a, a\in\mathbb{Z}$

for (y,x) in C, $y-x=-a \in \mathbb{Z}$

Thus, C is symmetric.

Combining the 3 properties, C is an equivalence relation.

Question 3 continues on the next page...

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.

Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

To show that $\tilde{f}$ is a function, we need to show that for any x, $\tilde{f}(x)$ has only one value.

Suppose we have $\tilde{f}([a]) = [b_1]$ and $\tilde{f}([a]) = [b_2]$.

Then $[b_1]$ is an equivalence class $[a+\frac{1}{2}]$

$[b_2]$ is also an equivalence class $[a+\frac{1}{2}]$

Thus, b_1 and b_2 are in the same class

$\therefore [b_1] = [b_2]$

$\therefore \tilde{f}$ is a function.

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

$g(a) = \frac{1}{b}$

There would exist $a_1 = 0.5$ and $a_2 = 1.5$ in the same class that has $b_1 = 2$ and $b_2 = \frac{2}{3}$ that are not in the same class.

$\frac{1-r^n}{1-r}$

full-binary tree

~~For~~ there is always ~~m~~

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?

$1+2+4+8=15$ $\frac{1-2^4}{1-2}$

Let the # of terminal vertices be a_n at each level n

~~for~~ $a_n = ma_{n-1}$

let the number of terminal vertices be a_n at level n.

$\because$ all parent node has m children

$a_n = m a_{n-1}$, so $a_n = m^n a_0$

now we have that. We have i internal vertices, and

$i = a_0 + a_1 + a_2 + \dots + a_{n-1} = a_0 + ma_{n-1} + m^2 a_0 + \dots + m^n a_0$

$\because a_0 = 1 \therefore i = \frac{1-m^n}{1-m}$

$i - mi = 1 - m^n$

$m^n = mi - i + 1$

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \le m^h$.

T has $mi-i+1$ terminal vertices and $mi+1$ vertices in total.

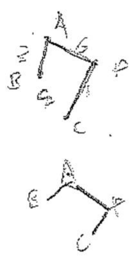
For h^{th} level, it has m^h terminal or less (when the tree has a $\boxed{\text{full } h^{th} \text{ level}}$, it is m^h)

$\therefore t \le m^h$

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

~~Suppose we've found a MST through Prim's algorithm. As all values are distinct, MST is unique as long as we choose the same vertex to start with.~~

~~If we choose a different vertex, we have to connect back to the vertex we originally used at some time, and choices made become the same as when we use this partical vertex.~~

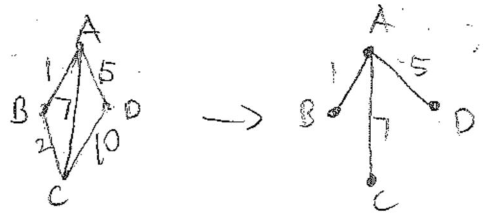


~~Thus, applying this to all vertices, they all generate the same ~~result~~ MST.~~

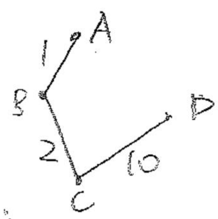
Suppose we get MST by eliminating edges in all cycles of G. Everytime we choose the smallest edge of a cycle. As all edges are distinct, the edge we choose to delete is always unique. Thus, we always have an unique MST.

Question 5 continues on the next page...

(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.



Using BSF, we get
weight = 1+5+7=13.



Using DSF, we get
weight = 1+2+10=13

6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

For a face to be formed, it has to have at least 3 edges, so 3F is the minimal number of edges boundry the faces.

each ~~edg~~ edge can be shared at most by 2 faces, so 2E is the maximum bounds that faces can have (repetition allowed).

Thus, $2E \geqslant 3F$.

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

For a simple graph, $F = E - V + 2$

$3F = 3E - 3V + 6$

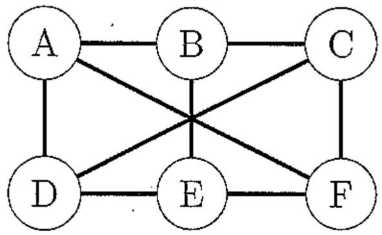
$\because 2E \geqslant 3F$

$2E \geqslant 3E - 3V + 6$

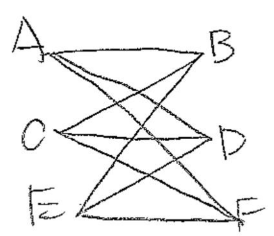
$3V - 6 \geqslant E$

Question 6 continues on the next page...

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.



It is not a planar because it can be transformed to $K_{3,3}$



As $K_{3,3}$ is not a planar graph, the graph above is not a planar either.

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Using inductive, when $n=1$

$7^1 - 1 = 6$ which is divisible by 6

Suppose the statement holds for $n=k$

$7^k - 1$ is divisible by 6

for $n=k+1$

$7^{k+1} - 1 = 7^k \cdot 7 - 1$

$= 7^k + 6 \cdot 7^k - 1$

$= 7^k - 1 + 6 \cdot 7^k$

$\because 7^k - 1 \bmod 6 = 0$ and $6 \cdot 7^k \bmod 6 = 0$

$\therefore (7^k - 1 + 6 \cdot 7^k) \bmod 6 = 0$, and the statement holds for $n = k+1$

Based on induction. $7^n - 1$ is divisible by 6 for $\forall n \geq 1$

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7.

$1 \bmod 7 = 1$

$11 \bmod 7 = 4$

$111 \bmod 7 = 2$

The modulus of $\sum_{i=0}^{n} 10^i$ by 7

is the modulus 10 times the previous number plus 1.

i.e. $a_n = (10 a_{n-1} + 1) \bmod 7$

we have $a_0 = 1$ $a_1 = 4$ $a_2 = 6$ $a_3 = 5$ $a_4 = 2$

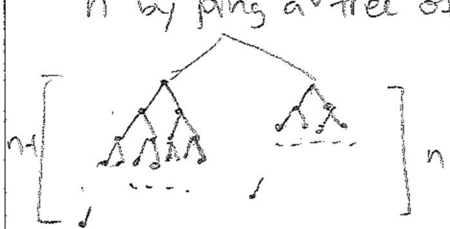
$a_5 = 0$.

so there is a number, 111111, divisible by 7.

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$

~~For a balanced binary trees,~~ we can obtain a tree of height n by joining a balanced binary tree of height $n-1$ (T_{n-1}) and BBT of height $n-2$ (T_{n-2}).



They differ by height of 1.
Because V_{n-1} and V_{n-2} are at minimum of their height, T_{n-1} and T_{n-2} form a V_n tree that is at minimum. Thus, adding a root that connects T_{n-1} and T_{n-2}, we have $V_n = V_{n-1} + V_{n-2} + 1$

(b) (3 points) Show that for $n \geq 0$, $v_n = F_{n+3} - 1$, where F_k is the k^{th} Fibonacci number.

1, 1, 2, 3, 5,

When the tree has height 0, $\rightarrow$ it has root $1 = F_{0+3} - 1$

When the tree has height 1, $\rightarrow$ it has $2 = F_4 - 1$

$\because V_n = V_{n-1} + V_{n-2} + 1$

$V_n = F_{n+2} - 1 + F_{n+1} - 1 + 1$

$= F_{n+2} + F_{n+1} - 2 + 1$

$= F_{n+3} - 1$

Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

$V_n = V_{n-1} + V_{n-2} + 1$

The homogeneous part of V_n is $V_n' = V_{n-1} + V_{n-2}$

$C_1 = -1,\ C_2 = -1$

$x^2 - x - 1 = 0 \quad x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$\therefore V_n' = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$

$V_n = V_n' + n$

$V_0 = 1 \quad V_2 = 2$

$\therefore \begin{cases} a+b=1 \\ \frac{1}{2}a + \frac{\sqrt{5}}{2}a + \frac{1}{2}b - \frac{\sqrt{5}}{2}b = 2 \end{cases} \rightarrow \begin{cases} a+b=1 \\ a-b = \frac{\frac{3}{2}}{\frac{\sqrt{5}}{2}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} \end{cases}$

$\therefore 2a = \frac{5+3\sqrt{5}}{5}$

$a = \frac{5+3\sqrt{5}}{10}$

$b = \frac{5-3\sqrt{5}}{10}$

$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{6+2\sqrt{5}}{4} = \frac{3}{2} + \frac{1}{2}\sqrt{5}$

Fabinacci sequence has $\Theta(\varphi^{n-1})$

So V_n has $\Theta(\varphi^{n+2})$

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

$3^n = (1+2)^n$

according to Binomial Theorem,

$$(1+2)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 2^i$$

$$= \sum_{i=0}^{n} \binom{n}{i} 2^i$$

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

$\binom{n+m}{r}$ is choosing r items from $(n+m)$ items.

$\binom{n}{i}\binom{m}{r-i}$ is choosing i items from n, then choosing $r-i$ items from m. The total chosen is r, but we restrict the distribution between two types a with m items and b with n items to be i and $r-i$.

Thus, $\binom{n}{r}\binom{m}{r-i}$ counts for choosing i from set a, which has n items, and $r-i$ from set b, which has m items. As we iterate i from 0 to r, we consider all the possibilities of distributing items chosen between these two categories. Thus, they add up to choosing r items from $(n+m)$ items, regardless of the category.

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