Math 61-1 Final exam

[redacted]

TOTAL POINTS

79 / 90

QUESTION 1

Multiple choice 10 pts

1.1 **0 / 2**

- **- 0 pts** Correct (c)
- ✓ **- 2 pts Incorrect**

1.2 **2 / 2**

- ✓ **- 0 pts Correct (a)**
- **- 2 pts** Incorrect

1.3 **2 / 2**

- ✓ **- 0 pts Correct (c)**
- **- 2 pts** Incorrect

1.4 **2 / 2**

- ✓ **- 0 pts Correct (b)**
- **- 2 pts** incorrect

1.5 **2 / 2**

- ✓ **- 0 pts Correct (a)**
- **- 2 pts** Incorrect

QUESTION 2

Short answer 10 pts

2.1 **2 / 2**

- ✓ **- 0 pts Correct ((-2)^100 + 3^100)**
- **- 1 pts** Almost correct (small arithmetic error in answer)
- **- 2 pts** Incorrect

2.2 **0 / 2**

- **- 0 pts** Correct (C(7,4)6!4!)
- **- 1 pts** Close
- ✓ **- 2 pts Incorrect**

2.3 **2 / 2**

- ✓ **- 0 pts Correct (24C4)**
- **- 1 pts** Close
- **- 2 pts** incorrect

2.4 **2 / 2**

- ✓ **- 0 pts Correct**
- **- 1 pts** Close (Three of four)
- **- 2 pts** Incorrect

2.5 **1 / 2**

- **- 0 pts** Correct (2^(n^2 - n) + 2^(n^2 +n / 2) - 2^(n^2 - n / 2))
- ✓ **- 1 pts Close**
- **- 2 pts** Incorrect

QUESTION 3

Equivalence relation 10 pts

3.1 it is an equivalence relation **4 / 4**

- ✓ **- 0 pts Correct**
- **- 1 pts** issue in transitivity
- **- 3 pts** misunderstanding of what relation is saying
- **- 4 pts** blank
- **- 2 pts** misunderstanding of symmetry
- **- 1 pts** the decimal thing isn't exactly right, e.g. -.3 is related to .7
- **- 0 pts** Click here to replace this description.
- **- 1 pts** issue with symmetry

3.2 defining a function **4 / 4**

- ✓ **- 0 pts Correct**
- **- 4 pts** blank
- **- 2 pts** need to prove uniqueness part of function
- **- 2 pts** missing existence part of function
- **- 1 pts** issue with uniqueness part of function

- **1 pts** need to consider different elements in the same equivalence class
- **1 pts** thing with decimals isn't quite right, for example -.3 and .7 are related
- **3 pts** big misunderstanding of the equivalence relation or function

3.3 a function that doesn't descend **0 / 2**

- **0 pts** Correct

✓ **- 2 pts your g is not a function**

- **1 pts** issue with justification
- **1 pts** your g does not work
- **2 pts** blank

QUESTION 4

m-ary tree 10 pts

4.1 number of internal vertices **3 / 5**

- **0 pts** Correct
- **1 pts** No/incorrect answer
- **4 pts** No/incorrect justification
- **2 pts** Didn't justify number of total vertices
- **3 pts** "Proof by example"
- **2 pts** Assumed every terminal vertex had the same height as the tree
- **5 pts** Nothing
- **1 pts** Forgot to account for root
- **2 pts** Didn't subtract off internal vertices

- 2 Point adjustment

Correct final answer but there are i internal vertices; you forgot to count the root vertex which is not a child, and your two mistakes cancelled out.

4.2 height **4 / 5**

- **0 pts** Correct
- **1 pts** No base case
- **1 pts** Didn't set up/invoke induction
- **1 pts** Backwards inductive step (didn't show inductive construction is exhaustive)
- **2 pts** Compared to complete tree without showing this case is extremal
- **3 pts** Assumed tree is complete / inductive construction forms complete trees from complete trees

✓ **- 1 pts Assumed all immediate subtrees have height h-1**

- **4 pts** "Proof by example"
- **5 pts** Nothing shown / Incorrect reasoning

QUESTION 5

spanning trees 10 pts

5.1 unique mst **5 / 6**

- **0 pts** Correct
- **3 pts** Appeal to Prim's or Kruskal's Algorithm (without proving it can generate any MST)
- **6 pts** No / Invalid reasoning

- 1 Point adjustment

You need to choose e to be the edge of smallest weight occuring in any cycle in G. Otherwise very well done!

5.2 non unique spanning tree **4 / 4**

✓ **- 0 pts Correct**

- **4 pts** Not an example
- **4 pts** Claimed no such graph exists
- **4 pts** Nothing

QUESTION 6

planar graphs 10 pts

6.1 2e > 3f **3 / 3**

✓ **+ 3 pts Correct**

+ **2 pts** >= 3 edges for each face
+ **1 pts** >= 3 edges for each face (w/ mistake)
+ **1 pts** <=2 faces for each edge
+ **0 pts** Incorrect

6.2 e<3v-6 **3 / 3**

✓ **+ 3 pts Correct**

+ **2 pts** Euler's formula

+ **1 pts** Correct application with (a)
+ **0 pts** Incorrect

6.3 nonplanar graph **4 / 4**

✓ + **4 pts Correct**
+ **3 pts** Isomorphic to K_3,3
+ **2 pts** Mistaken/missing ismorphism to K_3,3
+ **1 pts** E <= 2v-4 or 2E >= 4F
+ **1 pts** Other partial credit
+ **0 pts** Incorrect

QUESTION 7

10 pts

7.1 7^n-1 divisible by 6 **5 / 5**

✓ + **5 pts Correct**
+ **1 pts** Base case
+ **1 pts** Inductive hypothesis
+ **2 pts** factoring out a 7 in inductive step as (6+1) or adding/substracting 7
+ **1 pts** Conclusion
+ **0 pts** Incorrect

7.2 number with only 1s divisible by 7 **5 / 5**

✓ + **5 pts Correct**
+ **0 pts** Click here to replace this description.
+ **1 pts** Look at 8 consecutive terms
+ **1 pts** Pigeonhole remainder
+ **1 pts** 7 divides a number of the form 111..000...
+ **2 pts** This implies that 7 divides 10^k*11...
+ **1 pts** Unsuccessful attempt with substantial work

QUESTION 8

balanced binary trees 10 pts

8.1 4 / 4

✓ - **0 pts Correct**
- **2 pts** incomplete, need to describe how a height n minimal balanced binary tree is made out of ones of smaller height
- **3 pts** can't just do examples
- **4 pts** blank
- **1 pts** how are you adding in these trees/ vertices?
- **3 pts** can't do induction without using some properties of minimal balanced binary trees
- **4 pts** incorrect numbers/ equation

8.2 relationship to fibonacci numbers **3 / 3**

✓ - **0 pts Correct**
- **1.5 pts** that is not the recurrence/ equation for the fibonacci numbers/ minimal balanced binary trees
- **1 pts** you are assuming the desired conclusion
- **3 pts** blank
- **1.5 pts** need to use recurrence for fiboacci numbers
- **1.5 pts** missing inductive step
- **1 pts** the two recurrences aren't exactly the same, you need to account for this difference
- **0.5 pts** error in equations
- **1 pts** need to check initial conditions

8.3 Theta **3 / 3**

✓ - **0 pts Correct**
- **0.5 pts** need to account for other term in equation for fibonacci numbers (sometimes it is contributing something positive, something something negative)
- **2 pts** wrong formula for fibonacci numbers/ v_n
- **1 pts** issue with big O
- **1 pts** issue with omega
- **3 pts** blank/ no gradable work
- **1 pts** wrong equations/ issues with constants
- **2 pts** need to use equation for v_n/ Fibonacci numbers

QUESTION 9

binomial coefficients 10 pts

9.1 3^n **4 / 4**

✓ + **4 pts Correct**
+ **3 pts** Minor error
+ **2 pts** Binomial theorem
+ **1 pts** Attempted induction or counting argument
+ **0 pts** Incorret

9.2 vandermonde identity 6 / 6

✓ **+ 6 pts** **Correct**

+ 5 pts Minor errror

+ 3 pts One part of counting argument or (x+y)^n+m

+ 1 pts Attempted to use induction/binomial thrm/Pascal's identity

+ 0 pts Incorrect

Final

Name: [redacted]

Student ID: [redacted]

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$ $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

      (a) $\frac{(n+k)!}{k!n!}$

      (b) $\frac{(n+1)!}{k!(n+1-k)!}$

      (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$

      (d) none of the above

      $\frac{(n+1)!}{(k+1)!(n-k-1)!}$

   2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:

      (a) a height of $\geq \lg(n!)$

      (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)

      (c) a height of $O(\lg(n!))$

      (d) a height of $O(n \lg n)$

   3. If G is a graph with n vertices and $n-2$ edges, then:

      ~~(a)~~ G is a tree

      (b) G is connected

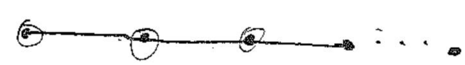
      (c) G is disconnected

      (d) G is simple

Question 1 continues on the next page...

4. Which of these graphs has an Euler cycle?
   (a) K_4
   (b) K_5
   (c) $K_{3,3}$
   (d) $K_{2,3}$

5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.
   (a) Less than or equal to n
   (b) More than n but less than or equal to $n^2/2$
   (c) More than $n^2/2$ but less than or equal to n^2
   (d) More than n^2



This is best case

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}?

$t^2 - t - 6 = 0$ $\quad t = 3, -2 \quad s_n = A3^n + B(-2)^n \quad A+B=2$

$(t-3)(t+2) = 0 \quad 3A - 2B = 1 \quad B = 1$

$\boxed{s_{100} = 3^{100} + (-2)^{100}} \quad 5A = 5 \Rightarrow A = 1$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

$\dfrac{7!\,P(8,4)}{11!}$

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

$\binom{20+5-1}{5-1} = \boxed{\binom{24}{4}}$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.

(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

$2^{\binom{n}{2}}$ = # of symetric $\quad 2^{P(n,2)}$ = # of reflexive

$\boxed{2^{\binom{n}{2}} + 2^{P(n,2)}}$

symmetric: choose 2 elements a,b, include $(a,b),(b,a)$ $\quad 2^{\binom{n}{2}}$

reflexive: include the n elements $(a,a), \dots$ then choose other elements $\quad 2^{P(n,2)}$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

- For all $x \in \mathbb{R}$, $x - x = 0$ which is an integer so $x - x \in \mathbb{Z}$ and $x C x$. Thus, C is reflexive.

- If $(x, y) \in C$, then $(x - y) \in \mathbb{Z}$ and $-(x-y) \in \mathbb{Z}$, so $y - x \in \mathbb{Z}$ and $(y, x) \in C$. Thus, for any $(x, y) \in C$, $(y, x) \in C$, so C is symmetric.

- If $(x, y), (y, z) \in C$, then we know that $x - y \in \mathbb{Z}$, and $y - z \in \mathbb{Z}$. The sum of two integers is also always an integer, so $(x-y) + (y-z) = x - z \in \mathbb{Z}$. Therefore $x C z$ and C is transitive.

- C is a reflexive, symmetric, and transitive relation, so it is an equivalence relation.

Question 3 continues on the next page...

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.

Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

For any $x_1, x_2 \in \tilde{\mathbb{R}}$, if $[x_1] = [x_2]$, then $\tilde{f}([x_1]) = [x_1 + \frac{1}{2}]$ and $\tilde{f}([x_2]) = [x_2 + \frac{1}{2}]$. We know that $x_1 + \frac{1}{2} - (x_2 + \frac{1}{2}) = 0 \in \mathbb{Z}$, so $[x_1 + \frac{1}{2}] = [x_2 + \frac{1}{2}]$. Also every $x \in \mathbb{R}$ has an equivalence class because C is an equivalence relation, so $\tilde{f}$ is a function. The reverse is also true: if $\tilde{f}([x_1]) \neq \tilde{f}([x_2])$ then we know that $\tilde{f}[x_1] - \tilde{f}[x_2] \notin \mathbb{Z}$ so $[x_1] \neq [x_2]$ because otherwise ↗ would not be true.

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

$g(x) = \frac{1}{x}$ $\qquad [\frac{1}{2}] = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$

$\tilde{g}([\frac{1}{2}]) = [2]$ $\qquad \tilde{g}([\frac{5}{2}]) = [\frac{2}{5}]$

$2 - \frac{2}{5} \notin \mathbb{Z}$, so $[2] \neq [\frac{2}{5}]$ and $\tilde{g}$ is not a function.

27
13

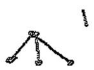
4 $1+3+9+$

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

   (a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?

4 9

4 + 5



1 1+2

3

1 + 2

27

each internal vertex has m children, so $m \cdot i$ children, where $\boxed{(i-1)}$ of those are internal vertices, so subtracting internal vertices $(i-1)$ from children $(m \cdot i)$:

$$m \cdot i - (i-1) = \boxed{i(m-1)+1}$$

This holds for the binary tree case as $t = i+1$

   (b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \leq m^h$.

This can be shown by induction on h.

Base Case: For a height 0 m-ary tree, there will be exactly 1 vertex, so $t \leq m^0 = 1$ is true.

Inductive Step: Assume the number of terminal vertices of any height h m-ary tree is $t \leq m^h$. We must show for a height $h+1$ m-ary tree, $t \leq m^{h+1}$.

- The number of terminal vertices is just the sum of all the terminal vertices of the individual height h child subtrees of the root. Thus, by the inductive hypothesis: $t \leq m \cdot m^h = m^{h+1}$ so $t \leq m^h$ for all $h \geq 0$ by mathematical induction. $\square$

T=4
i=3

3+1

4+5

T=9
i=4

5 + 7

5

16

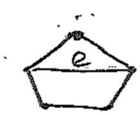
5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

Base Case: If G is a graph with 1 edge, then the graph is either 

, 

and each have a unique minimal spanning tree, so the base case holds.

Inductive Step: Assume all connected weighted graphs G with n edges with unique weights have a unique minimal spanning tree, we must show a graph with n+1 edges also has a unique minimal spanning tree. The graph G must have an edge incident on a vertex of degree 1 like so: or have a cycle:



In the first case, remove edge e and then use the inductive hypothsis to construct a minimal spanning tree of the n edge graph. Then add edge e to the tree and there is now a unique minimal spanning tree.

For the second case, simply remove e and use the inductive hypothesis to construct the unique minimal spanning tree.

We have prooved that a connected weighted graph G has a unique minimal spanning tree by the principle of mathematical induction. □

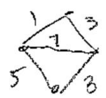
Question 5 continues on the next page...

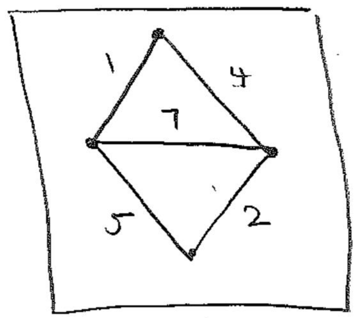
(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.

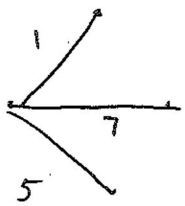
1 5

3

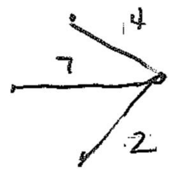
3







$1+5+7=13$



$4+2+7=13$

6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

Count number of boundary edges:

• each edge is a boundary to at most two faces: # boundaries $\leq 2E$

• Each face has at least 3 boundary edges

boundaries $\geq 3F$

$3F \leq$ # boundaries $\leq 2E$

$\boxed{3F \leq 2E}$

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

From part (a), $3F \leq 2E$. Eulers equation is $F-E+V=2$, so $F = E-V+2$

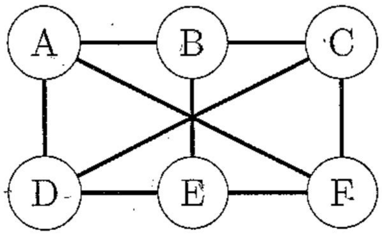
$3(E-V+2) \leq 2E$

$3E-3V+6 \leq 2E$

$\boxed{E \leq 3V-6}$

Question 6 continues on the next page...

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.



$E = 9 \qquad V = 6$

$F - E + V = 2$

$F = 9 - 6 + 2 = 5$

• Every cycle in this graph has length at least 4, so this implies that we should have

$2E \geq 4F$

$2(9) \geq 4(5)$

$18 \not\geq 20$, so it is not planar.

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Base case : If $n=1$, $7^1-1=6$ which is clearly divisible by 6, so the base case holds.

Inductive Step: Assume 7^n-1 is divisible by 6 for all n. We must show $7^{n+1}-1$ is also divisible by 6.

$$7^{n+1}-1 = 7 \cdot 7^n - 1 = 6 \cdot 7^n + (7^n - 1)$$

From the inductive hypothesis we know that 7^n-1 is divisible by 6. $6 \cdot 7^n$ is also clearly divisible by 6 because it has a factor of 6. The sum of two numbers divisible by 6 is also divisible by 6, so we have shown 7^n-1 is divisible by 6 for all $n \geq 1$ by mathematical induction. □

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7.

There are 7 possible remainders when dividing by 7, so considering the 8 numbers

1, 11, 111, 1111, 11111, 111111, 1111111, 11111111

by the pigeonhole principle, 2 distinct numbers from above have the same remainder when divided by 7.

By subtracting these two number, $x_1 - x_2$ is divisible by 7. $x_1 - x_2$ is of the form

$\underbrace{111\cdots1}_{n-m} \cdot 10^m$ if we ensure $x_1 > x_2$ and m is the number of digits in the smaller number x_2.

Thus we have a number $\underbrace{11\cdots1}_{n-m}$ divisible by 7. which is of the form $\sum_{i=0}^{n} 10^i$

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$

$n=1$ $v=2$ 

$n=2$ 

4

height $n-1$  height $n-2$

swap

V_n = root vertex + (# of vertices of right subtree) + (# of vertices of left subtree)

$= 1 +$ (min vertices for height $n-1$) $+$ (min vertices for height $n-2$)

$$V_n = \boxed{1 + V_{n-1} + V_{n-2}}$$

(b) (3 points) Show that for $n \geq 0$, $v_n = \cancel{F_{n+2}}$ $F_{n+3} - 1$, where F_k is the k^{th} Fibonacci number.

1 2 3 4 5
1, 1, 2, 3, 5

$n=0$: $V_0 = 1$, $F_3 - 1 = 2 - 1 = 1$ ✓

$n=1$: $V_1 = 2$, $F_4 - 1 = 3 - 1 = 2$ ✓

The base cases hold.

Inductive Step: Assume $V_k = F_{k+3} - 1$ for all $k \leq n$. We must show $V_{n+1} = F_{n+4} - 1$.

$V_{n+1} = 1 + V_n + V_{n-1}$ ← inductive hypothesis.

$= 1 + F_{n+3} - 1 + F_{n+2} - 1 = F_{n+4} - 1$

So $V_n = F_{n+3} - 1$ for all $n \geq 0$ by the principle of mathematical induction. □

Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

~~$$V_n = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n+3} - 1$$~~

~~$$\geq \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \left(\frac{1+\sqrt{5}}{2}\right)^{n+3}$$~~

~~because $\left(\frac{1+\sqrt{5}}{2}\right) > 1$~~

~~$$= \left(-\frac{2}{\sqrt{5}} - 1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+3}$$~~

~~$\geq$~~

~~because $\left(\frac{1+\sqrt{5}}{2}\right)^n$ is an increasing function~~

~~Thus~~

~~$$V_n = A\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} + B\left(\frac{1-\sqrt{5}}{2}\right)^{n+3} - 1$$~~

~~$\leq$~~

Let $\psi = \frac{1-\sqrt{5}}{2}$

$$V_n = F_{n+3} - 1 = F_{n+2} + F_{n+1} - 1$$

$$= \left(\frac{1}{\sqrt{5}}\right)\phi^{n+2} - \left(\frac{1}{\sqrt{5}}\right)\psi^{n+2} + \left(\frac{1}{\sqrt{5}}\right)\phi^{n+1} - \left(\frac{1}{\sqrt{5}}\right)\psi^{n+1} - 1$$

This is an exponential function where $\phi > \psi$ and ϕ^n is increasing, so the greatest power is ϕ^{n+2}, so $\boxed{V_n = \Theta(\phi^{n+2})}$

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1+\sqrt{5}}{2}\right) = 1 \qquad A = \frac{1 - B\left(\frac{1-\sqrt{5}}{2}\right)}{\left(\frac{1+\sqrt{5}}{2}\right)}$$

$$A\left(\frac{6+2\sqrt{5}}{4}\right) + B\left(\frac{6-2\sqrt{5}}{4}\right) = 1$$

$$\left(\frac{1 - B\left(\frac{1-\sqrt{5}}{2}\right)}{\left(\frac{1+\sqrt{5}}{2}\right)}\right)\left(\frac{6+2\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right) = 1$$

$$\left(1 - B\left(\frac{1-\sqrt{5}}{2}\right)\right)\left(\frac{3+\sqrt{5}}{1+\sqrt{5}}\right) + B\left(\frac{3-\sqrt{5}}{2}\right) = 1$$

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

Binomial theorem:

$$\sum_{i=0}^{n} 2^i \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} 2^i 1^{n-i} = (2+1)^n$$

$$= \boxed{3^n}$$

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

Choosing r items out of a group of n items and a group of m items is $\binom{n+m}{r}$. We can also choose i items from group of n, then the remaining (r-i) items from the group of m, so total for i items from group of n is $\binom{n}{i}\binom{m}{r-i}$. We want all possible numbers from group of n, so $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

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$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

$$= \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = 1 \checkmark$$

$$\frac{3+\sqrt{5}}{1+\sqrt{5}} - B\left(\frac{3-5-2\sqrt{5}}{2+2\sqrt{5}}\right) + B\left(\frac{3-\sqrt{5}}{2}\right) = 1$$

$$\frac{3+\sqrt{5}-3\sqrt{5}-5}{1-5} - B(-1) + B\left(\frac{3-\sqrt{5}}{2}\right) = 1$$

$$\frac{-2-2\sqrt{5}}{-4} + B\left(1 + \frac{3-\sqrt{5}}{2}\right) = 1$$

$$B\left(\frac{5-\sqrt{5}}{2}\right) = -\frac{1+\sqrt{5}}{2} \qquad B = \frac{-1-\sqrt{5}}{5-\sqrt{5}}$$

$$= \frac{-5-5\sqrt{5}-\sqrt{5}-5}{25-5}$$

$$= \frac{-10-6\sqrt{5}}{20} =$$