J. Cameron Math 61 Friday, March 22nd, 2019

Final

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Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

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Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$

   $\frac{n!\,(k+1)}{(k+1)!\,(n-k)!} + \frac{n!\,(n-k)}{(k+1)!\,(n-k)!}$

   $= \frac{n!\,(n\cancel{-k}\cancel{+k}+1)}{(k+1)!\,(n-k)!} = \frac{(n+1)!}{(k+1)!\,(n-k)!}$

   (a) $\frac{(n+k)!}{k!n!}$
   (b) $\frac{(n+1)!}{k!(n+1-k)!}$
   (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$
   (d) none of the above

2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:
   (a) a height of $\geq \lg(n!)$
   (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)
   (c) a height of $O(\lg(n!))$
   (~~d~~) a height of $O(n \lg n)$

3. If G is a graph with n vertices and $n-2$ edges, then:
   (a) G is a tree
   (b) G is connected
   (c) G is disconnected
   (d) G is simple

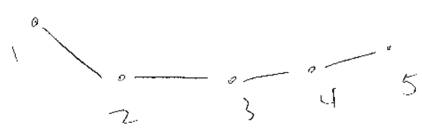
Question 1 continues on the next page...

Question 1 continued...

4. Which of these graphs has an Euler cycle?
   (a) K_4 $\delta=3$
   (b) K_5 $\delta=4$
   (c) $K_{3,3}$ $\delta=3$
   (d) $K_{2,3}$ $\delta=2,3$

5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.
   (a) Less than or equal to n
   (b) More than n but less than or equal to $n^2/2$
   (c) More than $n^2/2$ but less than or equal to n^2
   (d) More than n^2

5 vertices
4 edges



$t^2 = t + 6$ $\quad$ $t^2 - t - 6 = 0$ $\quad$ $(t-3)(t+2) = 0$

$S_n = A(3)^n + B(-2)^n$

$S_0 = A + B = 2$ $\quad$ $A = 1$

$S_1 = 3A - 2B = 1$ $\quad$ $B = 1$

$S_n = 3^n + (-2)^n$

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}?

$S_{100} = 3^{100} + (-2)^{100}$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

3 $\quad$ 2

BABA

ABAB

4P2

A_1 B_1 B_2 A_2

A_1 B_2 B_1 A_2

1 1 2 3 4 5 6 7

$\dfrac{7!\cdot\left(\binom{8}{4} - \binom{6}{2}\right)}{11}$

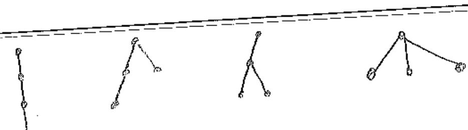
M_M_M_M_M_M_M_

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

1.1.1.1-

$\binom{24}{4}$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.



(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

$2^{\frac{n^2+n}{2}} + 2^{n^2-n} - 2^{\frac{n^2-n}{2}}$

symmetric: $2^{\frac{n^2+n}{2}}$ $\quad$ reflexive: 2^{n^2-n}

both: $2^{\frac{n^2-n}{2}}$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

Reflexive:

$\forall x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$ so $(x, x) \in C$, and C is reflexive.

Symmetry:

Let $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $x - y \in \mathbb{Z}$, i.e. $x - y$ is an integer n.

Then $y - x = -(x-y) = -n$ which is also an integer, so

$(y, x) \in C$ and C is symmetric.

Transitive:

Let $(x, y), (y, z) \in C$. So $x - y = n_1$ and $y - z = n_2$.

Then $x - z = (x-y) + (y-z) = n_1 + n_2$, and since both n_1 and n_2 are integers, $n_1 + n_2$ is also an integer, so $x - z \in \mathbb{Z} \Rightarrow$

$(x, z) \in C \Rightarrow C$ is transitive.

Since C is reflexive, symmetric, and transitive,

C is an equivalence relation.

Question 3 continues on the next page...

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.
Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

$f(x) = x + \frac{1}{2}$ $\quad$ $\tilde{f} : \tilde{R} \to \tilde{R} = \{([a],[b]) \in \tilde{R}\times\tilde{R} : f(a) = b\}$

$\forall [a] \in \tilde{R}$, $f(a) = a + \frac{1}{2}$ so there exists $([a], [a+\frac{1}{2}])$ in $\tilde{f}$.

Now suppose $([a],[b]) \in \tilde{f}$ and $([a],[c]) \in \tilde{f}$

$[a] = \{\alpha \in \mathbb{R} \mid \alpha = a + N, N \in \mathbb{Z}\}$

Since $f(x) = x + \frac{1}{2}$:

$([a],[b])$ & $([a],[c])$ must be of the form:

$([a+N_1], [a+N_1+\frac{1}{2}])$ and $([a+N_2], [a+N_2+\frac{1}{2}])$

and $(a+N_1+\frac{1}{2}, a+N_2+\frac{1}{2}) \in C$ so $[a+N_1+\frac{1}{2}] = [a+N_2+\frac{1}{2}]$ $\forall N_1, N_2 \in \mathbb{Z}$.

So if $([a],[b]) \in \tilde{f}$ and $([a],[c]) \in \tilde{f}$ then $[b] = [c]$ so $\tilde{f}$ is a function.

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

$C = \{(x,y) \mid x - y \in \mathbb{Z}\}$ $\quad$ $([a],[b])$ $\quad$ $([a],[c])$

Let $g(x) = \frac{x}{2}$

then $([5],[2.5]) \in \tilde{g}$ and $([10],[5]) \in \tilde{g}$

but $[5] = [10]$ (for relation C) yet $[2.5] \neq [5]$

so $([5],[2.5])$ & $([5],[5]) \in \tilde{g}$, so $\tilde{g}$ is not a function

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?

total vertices : $mi + 1$ root

internal vertices : i

terminal vertices: # total − # internal

$= \boxed{(m-1)i+1}$

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \le m^h$.

Proof by induction

Base case: a full m-ary tree of height $h=0$ has just 1 terminal vertex, and $1 \le m^0$ holds.

Induction Step:

Assume a full m-ary tree of height h has $t \le m^h$ terminal vertices. (Then the most terminal vertices it can have is m^h). Then the most a full m-ary tree of height h+1 can have is by adding m children at every terminal vertex of the height h tree, which means the new height (h+1) m-ary tree will have at most $mt \le m(m^h) = m^{h+1}$.

So we have shown that if a full m-ary tree of height h has $t \le m^h$ terminal vertices, then a full m-ary tree of height h+1 has $t' \le m^{h+1}$ terminal vertices, thus proved by induction.

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

Suppose T_1 and T_2 are distinct minimal spanning trees of G.

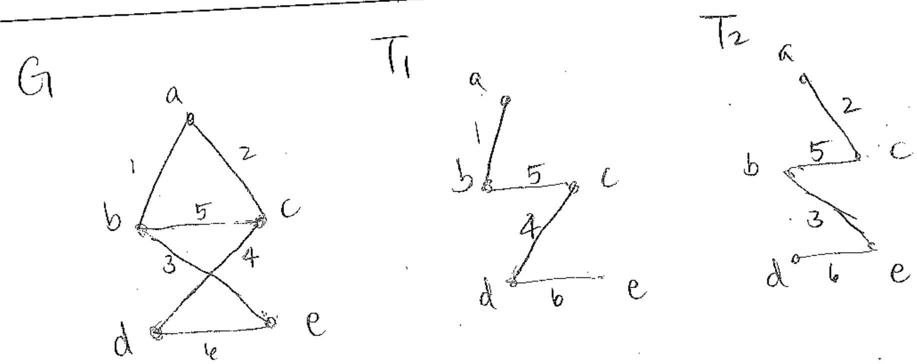
Let $e_1, e_2 \cdots e_{n-1}$ be the edges in T_1, and $e_1', e_2' \cdots e_{n-1}'$ be the edges in T_2.

WLOG suppose $e_1 < e_1'$ and that e_1 and e_1' share a vertex v. e.g. $e_1 = (u,v)$ and $e_1' = (v,w)$

Now we add e_1 to T_2, and T_2 now has a cycle that includes e_1 and e_1'. We remove e_1' from T_2 to get T_2', where T_2' is still connected & now acyclic so T_2' is a spanning tree. However $e_1 < e_1'$ so T_2' now has a smaller weight than T_2, which contradicts the assumption that T_2 is a distinct minimal spanning tree (different from T_1). So, by contradiction T_1 and T_2 must be the same.

Question 5 continues on the next page...

(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.



There are 2 distinct spanning trees, as shown above to the right: T_1 and T_2

T_1 has total weight $1+5+4+6 = 16$
T_2 has total weight $2+5+3+6 = 16$

6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

Since G is a simple graph, G has no loops (which allow for cycles of length 1) nor parallel edges (which allow for cycles of length 2). So, each cycle in G has minimum length 3. If G is a planar graph, then each face of G is bounded by a cycle of at least 3 edges, so there are at least 3 times as many edges as there are faces. However 1 edge is the boundary of 2 faces, so we overcount each edge twice, so instead we have $2E \geq 3F$.

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

We have shown $2E \geq 3F$.

Also by Euler's formula, $F = E - V + 2$

So $2E \geq 3(E - V + 2)$

$2E \geq 3E - 3V + 6$

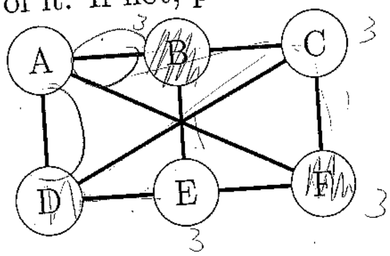
$3V - 6 \geq E$.

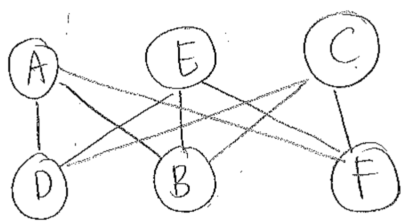
or

$E \leq 3V - 6$ ☒

Question 6 continues on the next page...

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.





It is not planar because, as shown above, the graph is homeomorphic to $K_{3,3}$ (actually we didn't have to do series reduction so it's also isomorphic). By the Kuratowski's Theorem, the graph is not planar.

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Proof by Induction

Base case: $n=1 \Rightarrow 7^n - 1 = 6$ which is divisible by 6

Induction Step:

Assume $7^n - 1$ is divisible by 6.

then $7^{n+1} - 1 = 7(7^n) - 1 = (6+1)(7^n) - 1$

$= 6 \cdot 7^n + 7^n - 1 = 6(7^n) + (7^n + 1)$

$6 \cdot 7^n$ is divisible by 6, and $7^n - 1$ is divisible by 6 by the inductive assumption. So, $6 \cdot 7^n + 7^n - 1 = 7^{n+1} - 1$ is divisible by 6, thus proved by induction.

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7.

$1001 = 7 \cdot 143$

$1001 \cdot 111 = 100100 + 10010 + 1001$

$= 111111 = 7(143)(111)$

Proof of existence:

$$\sum_{i=0}^{n} (7+3)^i = \sum_{i=0}^{n} \left(\sum_{k=0}^{i} 7^k 3^{i-k} \binom{i}{k} \right)$$

$$= \sum_{i=0}^{n} \left(7^0 3^i + \sum_{k=1}^{i} 7^k 3^{i-k} \binom{i}{k} \right) = \underbrace{\sum_{i=0}^{n} 3^i}_{\text{left term}} + \underbrace{\sum_{i=0}^{n} \sum_{k=1}^{i} \binom{i}{k} 7^k 3^{i-k}}_{\text{right term}}$$

The right term is always divisible by 7. The left term is a geometric series so $\sum_{i=0}^{n} 3^i = \frac{3^{n+1} - 1}{2}$ By the Fermat's little Theorem, $a^{p-1} \equiv 1 \pmod p$ for prime p, so we let $p=7$ and $a=3 \Rightarrow 3^6 \equiv 1 \pmod 7$

So for $n+1=6 \Rightarrow n=5$. $\Rightarrow 3^{n+1} - 1$ is divisible by 7. $\Rightarrow \sum_{i=0}^{n} 3^i$ is divisible by 7

$$\Rightarrow \sum_{i=0}^{n} 3^i + \sum_{i=0}^{n} \sum_{k=1}^{i} \binom{i}{k} 7^k 3^{i-k} = \sum_{i=0}^{n} 10^i \text{ is divisible by } 7.$$

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$

For a balanced binary tree of height n, at least one of its left/right subtrees must have a height of $(n-1)$. Then by the definition of "balanced", the other subtree must have a height of $(n-2)$ or $(n-1)$ (it can't be n because then the original tree wouldn't have height n). To achieve the minimum # of vertices, we choose $h = n-2$. WLOG say the left subtree has $h_l = n-1$ and the right subtree has height $h_r = n-2$. Then the number of minimum vertices in the left and right subtrees are then V_{n-1} and V_{n-2}, so $V_n = V_{n-1} + V_{n-2} + 1$ (+1 for the root)

(b) (3 points) Show that for $n \geq 0$, $\cancel{v_n = F_{n+2}}$, where F_k is the k^{th} Fibonacci number. $V_n = F_{n+3} - 1$

$V_n = f_n - 1$

F_0	0
F_1	1
F_2	1
F_3	2
F_4	3
F_5	5
F_6	8

$V_n = V_{n-1} + V_{n-2} + 1$

A constant particular solution of V_n is:

$V_p = V_p + V_p + 1 \Rightarrow V_p = -1$

The homogeneous solution of V_n is:

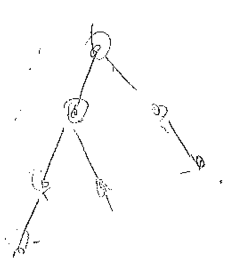
$V_n^h = V_{n-1}^h + V_{n-2}^h$ and $V_n = V_n^p + V_n^h \Rightarrow V_n^h = V_n - V_n^p = V_n + 1$

$V_2^h = V_2 + 1 = 4 + 1 = 5$ $\quad V_3^h = V_3 + 1 = 7 + 1 = 8$

$F_5 = V_2^h = 5$ and $F_6 = V_3^h = 8$ and $F_{n+3} = F_{n+2} + F_{n+1}$

since F_{n+3} & V_n^h have the same recurrence relation and initial conditions, $F_{n+3} = V_n^h$

So $V_n = V_n^p + V_n^h = F_{n+3} - 1$



Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

$$F_n = A\left(\frac{1+\sqrt5}{2}\right)^n + B\left(\frac{1-\sqrt5}{2}\right)^n$$

$$F_0 = A+B = 0$$

$$F_1 = \frac{A+A\sqrt5}{2} + \frac{B-B\sqrt5}{2} = 1$$

$$\frac{\cancel{A+B}^{\,0}}{\cancel{2}} + \frac{(A-B)\sqrt5}{2} = 1$$

$$(A-B)\sqrt5 = 2$$

$$2A\sqrt5 = 2$$

$$A = \frac{1}{\sqrt5}$$

$$F_n = \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n$$

$$V_n = F_{n+3} - 1$$

$$= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{n+3} - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^{n+3} - 1$$

$$= \left(\frac{1+\sqrt5}{2}\right)^{n+2}\left(\frac{\sqrt5+5}{10}\right) - \left(\frac{\sqrt5-5}{10}\right)\left(\frac{1-\sqrt5}{2}\right)^{n+2} - 1$$

$$= \left(\frac{\sqrt5+5}{10}\right)\left(\frac{1+\sqrt5}{2}\right)^{n+2} + \left(\frac{5-\sqrt5}{10}\right)\left(\frac{1-\sqrt5}{2}\right)^{n+2} - 1$$

$$\leq \left(\frac{\sqrt5+5}{10}\right)\left(\frac{1+\sqrt5}{2}\right)^{n+2} + \left(\frac{5-\sqrt5}{10}\right)\left(\frac{1+\sqrt5}{2}\right)^{n+2}$$

$$= \left(\frac{1+\sqrt5}{2}\right)^{n+2} = O(\phi^{n+2})$$

$$V_n = F_{n+3} - 1$$

$$= \left(\frac{\sqrt5+5}{10}\right)\left(\frac{1+\sqrt5}{2}\right)^{n+2} + \left(\frac{5-\sqrt5}{10}\right)\left(\frac{1-\sqrt5}{2}\right)^{n+2} - 1$$

$$\geq \left(\frac{\sqrt5+5}{10}\right)\left(\frac{1+\sqrt5}{2}\right)^{n+2} - \left(\frac{5-\sqrt5}{10}\right)(1)^{n+2} - 1$$

$$\geq \left(\frac{\sqrt5+5}{10}\right)\left(\frac{1+\sqrt5}{2}\right)^{n+2} - \left(\frac{5-\sqrt5}{10}\right)\left(\frac{1+\sqrt5}{2}\right)^{n+2} - 1$$

$$= \frac{\sqrt5}{5}\left(\frac{1+\sqrt5}{2}\right)^{n+2} - 1 = \Omega(\phi^{n+2})$$

$$\frac{1-\sqrt5}{2} \geq \frac{1-3}{2} = -1$$

$$-1 \geq -\frac{1+\sqrt5}{2}$$

Since $V_n = O(\phi^{n+2}) = \Omega(\phi^{n+2})$

then $V_n = \Theta(\phi^{n+2})$

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

By the binomial theorem, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Let $a=2$ and $b=1$

$\Rightarrow \quad (2+1)^n = \sum_{i=0}^{n} \binom{n}{i} 2^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i} 2^i$

So. $3^n = \sum_{i=0}^{n} \binom{n}{i} 2^i$

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$

We can give a combinatorial argument: There are $n+m$ distinct items, split into two groups, one with n items & one with m items. We want to choose r items which can be done in $\binom{n+m}{r}$ ways. We can also choose r items by choosing i items from the group of n items and then $r-i$ items from the group of m items. There are $\binom{n}{i}\binom{m}{r-i}$ ways to do this, and i can be any number from 0 to r. So $\binom{n+m}{r}$ is also equal to $\sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

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