Final

Name:

Student ID:

Section:

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.
Please get out your id and be ready to show it during the exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$

      (a) $\frac{(n+k)!}{k!n!}$

      (b) $\frac{(n+1)!}{k!(n+1-k)!}$

      (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$

      (d) none of the above

$\binom{n}{k} + \binom{n}{k+1}$

B

$\binom{n+1}{k} = \frac{(n+1)!}{k!\,(n+1-k)!}$

$\binom{2}{1} + \binom{2}{2} = 3 \quad \binom{3}{1} = 3$

   2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:

      (a) a height of $\geq \lg(n!)$

      (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)

      (c) a height of $O(\lg(n!))$

      (d) a height of $O(n \lg n)$

n! possibilities

   3. If G is a graph with n vertices and $n-2$ edges, then:

      (a) G is a tree

      (b) G is connected

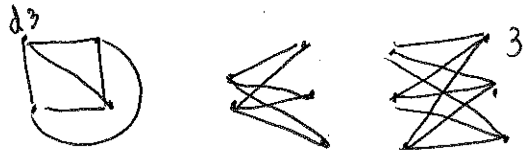
      (c) G is disconnected

      (d) G is simple

Question 1 continues on the next page...

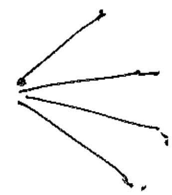
estion 1 continued...

4. Which of these graphs has an Euler cycle?
   (a) K_4
   (b) K_5
   (c) $K_{3,3}$
   (d) $K_{2,3}$



5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.
   (a) Less than or equal to n
   (b) More than n but less than or equal to $n^2/2$
   (c) More than $n^2/2$ but less than or equal to n^2
   (d) More than n^2

n - 1



2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}?

$t^2 - t - 6 = 0$
$(t-3)(t+2) = 0$
$\Rightarrow t = 3, -2$

$S_n = b(3)^n + d(-2)^n$
$\begin{cases} b + d = 2 \Rightarrow d = 2 - b \\ 3b - 2d = 1 \Rightarrow 3b - 2(2-b) = 1 \end{cases}$
$5b - 4 = 1 \Rightarrow b = 1, d = 1$

$\Rightarrow S_n = 3^n + (-2)^n$
$S_{100} = 3^{100} + (-2)^{100}$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

$\dfrac{7!\binom{8}{4}}{11}$

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

$\binom{20+5-1}{5-1} = \binom{24}{4}$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.

(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

~~pairs: $n^2 - n$~~
~~reflexive: 2^{n^2-n}~~

$2^{\frac{n^2-n}{2}}$

3. Consider the relation on the real numbers defined by $C = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

Reflexivity: For $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$, so ~~xxx~~ xCx, and C is reflexive

Symmetry: For $x, y \in \mathbb{R}$, if xCy, then $x - y \in \mathbb{Z}$, so $-(x-y) = y - x \in \mathbb{Z}$, so yCx. Since if xCy, then yCx, C is Symmetric.

Transitivity: For $x, y, z \in \mathbb{R}$, if xCy and yCz, then $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$, so $(x-y)+(y-z) = x - z \in \mathbb{Z}$, so xCz. Since if xCy and yCz then xCz, C is transitive.

C is reflexive, symmetric, and transitive, so C is an equivalence relation.

Question 3 continues on the next page...

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.
Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

~~Let $x \in \mathbb{R}$, then $f(x) = x + \frac{1}{2}$, so for x, there always exists a $f(x)$. Next, let $x_1, x_2 \in \mathbb{R}$. If $x_1 = x_2$, we have $x_1 + \frac{1}{2} = x_2 + \frac{1}{2}$, or $f(x_1) = f(x_2)$.~~

Let $x \in [a]$. Since $x \in [a]$, there is some y for x, namely $y = x + \frac{1}{2}$. So for $x \in [a]$, there is some $y \in [b]$ such that $f(x) = y$.
Let $x_1, x_2 \in [a]$, and suppose $x_1 = x_2$. Then $y_1 = x_1 + 1/2$, $y_2 = x_2 + 1/2$, but since $x_1 = x_2$, $y_1 = y_2$. Thu, for any $x_1 = x_2 \in [a]$, $f(x_1) = f(x_2) = y$, $y \in [b]$.

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

$g(x) = 1$ since $1 \in [a]$ could be mapped to multiple numbers in $[b]$

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?

If we have i internal vertices, then all i vertices have m children, and all of these vertices are themselves children except for the root vertex. So the total number of vertices is $mi+1$, and the number of terminal vertices is $mi+1-i$, or $(m-1)i+1$

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \leq m^h$.

Proof by induction on height h

Base case: $h=0$ $\quad t=1,\ m^h=1 \quad 1\leq 1$ ✓

Suppose $t \leq m^h$ for trees of heights less than h.

let T be a full m-ary tree of height h with t terminal vertices.

Then the root vertex of T has m children, ~~[illegible]~~ let $T_1, T_2, \ldots T_m$ be the subtrees rooted at the root vertex's children. Each subtree has heights $h_1, h_2, \ldots h_m < h$.

Then $t = \underbrace{t_1 + \cdots + t_m}_{\text{terminal vertices of each } T_1 \ldots T_m} \leq m^{h-1} + \ldots + m^{h-1} = m(m^{h-1}) = m^h$.

So $t \leq m^h$.

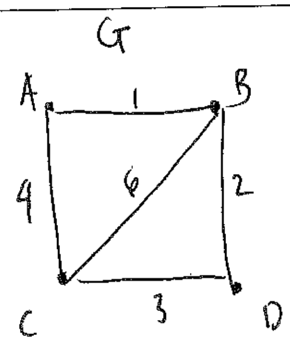
5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

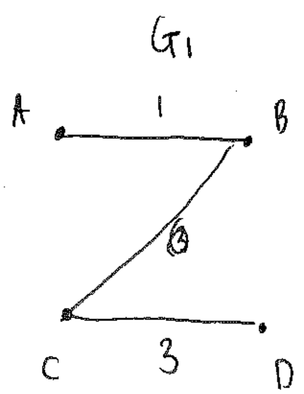
Suppose G does not have a unique minimal spanning tree. Then there is at least one edge that can be replaced by one or more edges to create a different minimal spanning tree. If this edge can be replaced by a single edge, then these two edges have the same weights, which is a contradiction. If this edge can be replaced by multiple edges, ~~the single a minimal spanning tree has n-1 vertices,~~ then ~~some~~ at least one edge is unnecessary and can be removed to lower the ~~w~~ total weight of the graph. This is also a contradiction, since we stated that this tree was a minimal spanning tree.

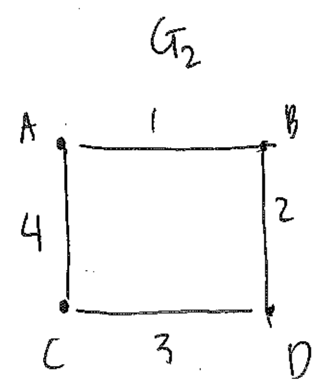
Thus, if G is a connected weighted graph where all edges have distinct weights, then G has a unique minimal Spanning tree.

Question 5 continues on the next page. . .

(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.







Both G_1 and G_2 have total weight 10

6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

An edge can bound at most 2 faces in a planar graph, and a ~~cycle~~ face must consist of at least 3 edges.

$2E \geq 3F$ follows.

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

$F = E - V + 2$

$2E \geq 3F$

$\frac{2E}{3} \geq F$

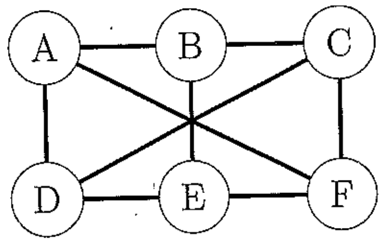
$\frac{2E}{3} \geq E - V + 2$

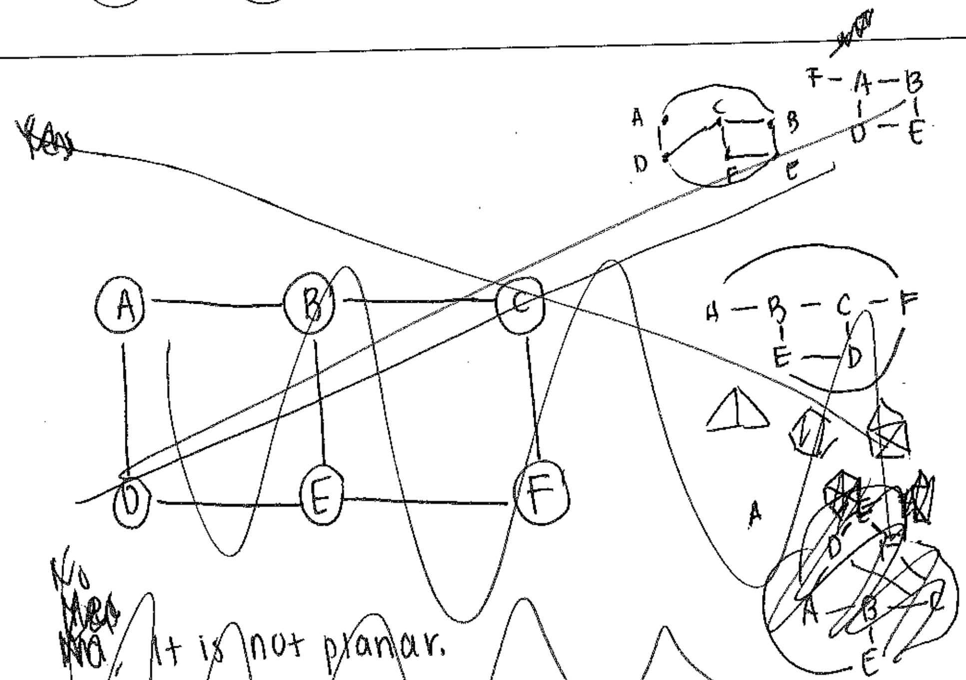
$\Rightarrow 2E \geq 3E - 3V + 6$

$\Rightarrow E \leq 3V - 6$

Question 6 continues on the next page...

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.

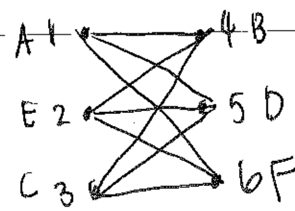




No, It is not planar.

$E = 9 \quad V = 6 \quad F = E - V + 2 = 5$

$2(9) \geq 3(5)$



No, it is not planar since it is isomorphic to $K_{3,3}$.

$\Leftarrow$ $f(A) = 1$ $\quad f(E) = 2$

$f(B) = 4$ $\quad f(F) = 6$

$f(C) = 3$

$f(D) = 5$

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Proof by Induction on n

Base case: $n=1$, $7^1 - 6 = 6$ 6 is divisible by 6 ✓

Suppose $7^n - 1$ is divisible by 6 for n.

$7^{n+1} - 1 = 7 \cdot 7^n - 1 = (7^n - 1) \cdot 7 + 6$

$7^n - 1$ is divisible by 6 by the inductive hypothesis, so $7(7^n - 1)$ is divisible by 6. 6 is also divisible by 6. The sum of two numbers each divisible by 6 is divisible by 6, so $(7^n - 1) \cdot 7 + 6 = 7^{n+1} - 1$ is divisible by 6.

So for $n \geq 1$, $7^n - 1$ is divisible by 6.

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7. $\underbrace{11}_{1\text{'s}}$

Suppose we take the first 8 number of this form,

1 11 111 1111 11111 111111 1111111 11111111.

For the remainder of a number divided by 7, there are 7 possibilities, 0–6, so by the Pigeonhole Principle, two of the numbers above have the same remainder.

Then subtracting the smaller number from the bigger one, we have a number of the form $\underbrace{111\ldots1}_{1\text{'s}}\underbrace{00\ldots0}_{0\text{'s}}$ that is divisible by 7.

We can rewrite this number as $\underbrace{111\ldots1}_{1\text{'s}} \times 10^n$ where n is the number of digits in the smaller number. Since 10^n is not divisible by 7, the number formed by the 1's $\underbrace{111\ldots1}_{1\text{'s}}$ must be divisible by 7.

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$

We can form a balanced binary tree by creating a new vertex a. Then ~~the~~ one of the subtrees of a is the balanced binary tree of height n-1, and the other is the balanced binary tree of height n-2. Then the heights of the left and right subtrees of a differ by 1. So V_n satisfies $V_n = \underbrace{V_{n-1}}_{\text{subtree height } n-1} + \underbrace{V_{n-2}}_{\text{height } n-2} + \underbrace{1}_{\text{new root vertex}}$.

(b) (3 points) Show that for $n \geq 0$, $v_n = \cancel{F_{n+2}}$ $F_{n+3}-1$, where F_k is the k^{th} Fibonacci number.

$V_n = V_{n-1} + V_{n-2} + 1$

$\cancel{F_{n+3-1} = F_{n+3-1-1} + F_{n+3-1-2} + 1}$

$\cancel{\Rightarrow F_{n+2} = F_n}$

$F_{n+3} - 1 = F_{n+3-1} + F_{n+3-2} + 1 - 1$

$\Rightarrow F_{n+3} - 1 = F_{n+2} + F_{n+1}$

$V_0 = 1,\ V_1 = 2,\ F_3 - 1 = 1,\ F_4 - 1 = 2$

So initial conditions $V_0 = F_3 - 1$, $V_1 = F_4 - 1$ match and recurrence pattern is the same, so $V_n = F_{n+3} - 1$.

Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

$V_n = F_{n+3} - 1$

We have before proved the complexity of F.

$F_n = F_{n-1} + F_{n-2}$

$t^2 - t - 1 = 0 \Rightarrow t = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$F_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$

Um:

$F_n = \theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$

$V_n = F_{n+3} - 1 = \theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}\right) = \theta(\phi^{n+2})$

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

$$\sum_{i=0}^{n} a^i b^{n-i} \binom{n}{i} = (a+b)^n$$

So $$\sum_{i=0}^{n} 2^i (1)^{n-i} \binom{n}{i} = (2+1)^n$$

or $$\sum_{i=0}^{n} 2i \binom{n}{i} = 3^n$$

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

$\binom{n}{i}$ is choosing i things from n items, and $\binom{m}{r-i}$ is choosing r-i items from m items. So if we wanted to choose r items in total, $\binom{n}{i}\binom{m}{r-i}$ is the number of ways of choosing i from a collection of n items and r-i from a different collection of m items. Then $\sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$ is the total number of ways of choosing r items from the two collections.

$\binom{n+m}{r}$ is the number of ways of choosing r items from n+m items.

This page has been left intentionally blank. You may use it as scratch paper. It will not be graded unless indicated very clearly here and next to the relevant question.