Problem 1. (20 points)

Compute the probability that 4-subset A of $\{1, 2, \ldots, 10\}$ satisfies:

a) A has no odd numbers,

b) A has at least one number ≤ 3,

c) A contains 1 but not 7.

d) the smallest number in A is divisible by 3

1 2 3 4 5 6 7 8 9 10

a) $\frac{{}_5C_4}{{}_{10}C_4}$ $\boxed{\frac{{}_5C_4}{{}_{10}C_4}}$ ✓

b) $\frac{{}_{10}C_4 - {}_7C_4}{{}_{10}C_4}$ $\boxed{\frac{{}_{10}C_4 - {}_7C_4}{{}_{10}C_4}}$ ✓

c) $\frac{1 \cdot {}_8C_3}{{}_{10}C_4}$ $\boxed{\frac{{}_8C_3}{{}_{10}C_4}}$ ✓

d) $\frac{{}_{10}C_4 - {}_7C_4}{{}_{10}C_4}$ $\boxed{\frac{{}_{10}C_4 - {}_7C_4}{{}_{10}C_4}}$ ✗ $-4.$

Problem 2. (20 points)

Let $X = \mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all non-negative integers. For each of the following functions $f : X \to X$ decide whether they are injective, surjective, bijective:

a) $f(x) = x + 1$

b) $f(x) = x^2 - 1$

c) $f(x) = 2x$

d) $f(x) = (x^2 + 2x)/(x + 2)$

$-1 \to 0$
$0 \to -1$
$1 \to 0$

a) $\boxed{\text{bijective}}$ 0

b) $\boxed{\text{injective}}$ 2

c) $\boxed{\text{injective}}$ 3

d) $$\frac{x^2+2x}{x+2} = \frac{\cancel{(x+2)}(x)}{\cancel{x+2}} = f(x)$$

-2.

$\boxed{\text{bijective}}$ 4

$1111 = 101$

$11 = 1$

Problem 3. (15 points)

Let $a_n = 1111\cdots 1$ (n ones). Suppose a_k is divisible by 97. Use induction to show that $a_{k\cdot n} = 0 \bmod 97$, for all $n \geq 1$.

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D: induction

Base: $a_k = 0 \bmod 97$

Inductive Step: $a_{k\cdot n} = a_{k(n+1)} = 0 \bmod 97$

$a_{k\cdot n} = 0 \bmod 97 = a_{k(n+1)} = 0 \bmod 97$

$= a_k$

$\frac{5!}{3!2!}$ $\frac{3!}{2!1!}$ $\frac{4!}{2!2!}$

Problem 4. (15 points)

Find closed formulas for the following sequences :

1 1 2 3 5 8

a) $4, 4, 6, 8, 12, 18, 28, 42, 70, 112, \ldots$

b) $a_1 = 1,\ a_{n+1} = a_n \cdot \binom{n+1}{2}$ $a_1 = 1 \Big\{ a_2 = 1\binom{2}{2} = 3 \Big\} a_3 = 3\binom{4}{2} = 18$

c) $a_1 = 1,\ a_2 = 1,\ a_{n+1} = a_{n-1} - a_n$ for $n \geq 2$.

Note: you can express a_n in terms of Fibonacci numbers F_n. $a_4 = 18\binom{5}{2} = 180$

a) $\boxed{a_n = 2F_N + 2}$ Ok

1 3 18 180

~~2 15 162~~

b) $\boxed{a_n = F_N^2}$

3 6 10

3 4

c) $\boxed{a_n = F_N^2}$

$a_1 = 1$ $a_2 = 1$ $a_3 = a_1 - a_2 = 1 - 1 = 0$

$a_4 = a_2 - a_3 = 1 - 0 = 1$ $F_N \% 2$

$a_5 = a_3 - a_4 = 0 - 1 = -1$

$a_6 = a_4 - a_5 = 1 - -1 = 2$

$a_7 = a_5 - a_6 = -1 - -2 = 1$

$a_8 = a_6 - a_7 = 2 - 1 = 1$

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

T F (1) The number of functions from $\{A, B, C, D\}$ to $\{1, 2, 3\}$ is equal to 4^3.

T F (2) The sequence $1, 3/2, 5/3, 7/6, 9/8, \ldots$ is increasing.

T F (3) The sequence $-1, -2, -3, -4, \ldots$ is non-increasing.

T F (4) There are 4 anagrams of the word MAMA. $\frac{4!}{2!2!} = \frac{12}{2!} = 6$

T F (5) There are infinitely many Fibonacci numbers which are divisible by 3.

T F (6) The number of permutations of $\{1, 2, 3, 4, 5\}$ is smaller than 123. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$4 \cdot 6$

$24 \cdot 5$

T F (7) The number of 3-permutations of $\{1, 2, 3, 4, 5, 6\}$ is equal to $\binom{6}{3}$.

T F (8) The number of 3-subsets of $\{1, 2, 3, 4\}$ is equal to 4. $\binom{4}{3}$ $\frac{4!}{3!1!}$

T F (9) The number of permutations of $\{1, 2, \ldots, n\}$ which have n preceding $n - 1$ (not necessarily immediately) is equal to $n!/2$

T F (10) For every $A, B \subset \{1, 2, \ldots, 12\}$ we have $|A \cap B| < |A \cup B|$. $A = 1$ $B = 1$

T F (11) For all $n \geq 1$, we have

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}.$$

T F (12) The number of grid walks from $(0, 0)$ to $(10, 10)$ going through $(3, 7)$ is equal to $\binom{10}{3}^2$. $\cancel{\binom{20}{10}} - \binom{10}{7}\binom{10}{3}$

T F (13) The number of grid walks from $(0, 0)$ to $(10, 10)$ avoiding $(10, 0)$ and $(0, 10)$ is equal to $\frac{1}{2}\binom{20}{10}$. $\binom{20}{10} - \binom{10}{0} - \binom{10}{0}$

T F (14) The number of anagrams of MISSISSIPPI which begin with M is greater than the number of anagrams which begin with S.

T F (15) The following parabolas are drawn in the plane:

$$y = x^2 - n^2 x - n^3, \qquad n = 1, \ldots, 12.$$

Then the regions of the plane separated by these parabolas can be colored with two colors in such a way that no two adjacent regions have the same color.