MIDTERM 1 (MATH 61, SPRING 2017)

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Math 61 Section: 1C

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The rules:
You MUST simplify completely and BOX all answers with an INK PEN.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: those caught violating the rules get automatic 10% score deduction.

Points:

1	20
2	20
3	3
4	0
5	30

Total: 73 **(out of 100)**

Problem 1. (20 points)

Compute the probability that 4-subset A of $\{1, 2, \ldots, 10\}$ satisfies:

a) A has no odd numbers.

b) A has at least one number ≤ 3.

c) A contains 1 but not 7.

d) the smallest number in A is divisible by 3

probability = $\dfrac{\text{subsets that satisfy}}{\text{~~all subsets~~}}$

a) 2 4 6 8 10 5 even numbers

$\{1 \ldots 10\} = \binom{n}{k}$ total k-subsets of $n = \binom{10}{4}$

$$\boxed{\frac{\binom{5}{4}}{\binom{10}{4}}}$$ ✓.

b) $$\boxed{\frac{\binom{10}{4}-\binom{7}{4}}{\binom{10}{4}}}$$ ✓.

$\binom{10}{4}$ all numbers combinations $\binom{7}{4}$ all numbers combinations without 3 or less

c) $$\boxed{\frac{\binom{8}{3}}{\binom{10}{4}}}$$ ✓.

$\binom{8}{3}$ ways to have 1 and not 7 since 1 is chosen and 1 and 7 cannot be chosen

d) smallest # in a is 3 or 6 ~~or 9~~ ← will not be smallest

~~1 instance where~~

6 7 8 9 ~~3 ways~~
6 7 8 10 4 ways
6 8 9 10
6 7 9 10

$\binom{4}{3}$ $\frac{4!}{3!\,1!} = 4$

$$\boxed{\frac{\binom{7}{3}+4}{\binom{10}{4}}}$$ ✓.

3 4 5 6 7 8 9 10 $\binom{7}{3}$

Problem 2. (20 points)

Let $X = \mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all non-negative integers. For each of the following functions $f : X \to X$ decide whether they are injective, surjective, bijective:

a) $f(x) = x + 1$

b) $f(x) = x^2 - 1$

c) $f(x) = 2x$

d) $f(x) = (x^2 + 2x)/(x + 2)$

a) $y = x + 1$

$\boxed{\text{injective ✓}}$ each x is only mapped to once

5

surjective X no way to map to 0

bijective X not injective ~~of~~ and surjective

b) $y = x^2 - 1$

5

$0 \to -1$? not a valid function so not any of them

~~even if it was valid, still none of them.~~

if it were valid than injection

c) $y = 2x$

$0 \to 0$

$1 \to 2$

$2 \to 4$

$3 \to 6$

5

$\boxed{\text{injective}}$ each x is mapped once max

not surjective, not each y is mapped

not bijective, not i or s

d) $y = \frac{x\cancel{(x+2)}}{\cancel{x+2}}$

5

injection, surjection, bijection

each x once

each x present

i and s

Problem 3. (15 points)

Let $a_n = 1111\cdots1$ (n ones). Suppose a_k is divisible by 97. Use induction to show that $a_{k\cdot n} = 0 \bmod 97$, for all $n \geq 1$.

base case

true, not needed

$a_1 = 1$ $a_k \bmod 97 = 0$

3/15

$a_{1\cdot k} = a_k \bmod 97 = 0$ ✓

inductive step

$a_n = \underbrace{111\ldots11}_{n \text{ ones}}$ $a_k \bmod 97 = 0$

assume $a_{nk} = a_k \bmod 97 = 0$

then $a_{(n+1)k} = a_k \bmod 97 = 0$

$a_{nk+k} = a_k \bmod 97 = 0$

$97 \mid a_{nk+k} - a_k$ $\boxed{\checkmark}$

Problem 4. (15 points)

Find closed formulas for the following sequences :

a) 4, 4, 6, 8, 12, 18, 28, ~~42~~ 44, 70, 112. . .

b) $a_1 = 1$, $a_{n+1} = a_n \cdot \binom{n+1}{2}$

c) $a_1 = 1$, $a_2 = 1$, $a_{n+1} = a_{n-1} - a_n$ for $n \geq 2$.

Note: you can express a_n in terms of Fibonacci numbers F_n

1 1 2 3 5
4 4 8 12 20
2 2 4 6 10

a) 4 4 6 8 12 18 28 44 70 112
2 2 4 6 10 16 26 42

$a_n = \cancel{4+F_n \cdot 2}$ $\boxed{\cancel{F_n \cdot 2}}$ $a_n = \boxed{F_n \cdot 2 - F_{n-1}}$

b) $\boxed{a_n = \left(a_{n-1}\left(\frac{(n+1)!}{2!(n+1-2)!}\right)\right)}$

c) ~~1 1 0 1 2 1 1 2 1~~

$a_3 = a_1 - a_2$ 0
$a_4 = a_2 - a_3$ 1
$a_5 = a_3 - a_4$ -1
a_6 4-3 1

$a_n = \boxed{\cos\cancel{(\pi n)}}$

$\boxed{= \cos\left(\cancel{\frac{\pi(n+1)}{2}}\right)}$

$\boxed{\cos\left(\frac{\pi}{2}(n-1)\right)}$

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

T F (1) The number of functions from $\{A, B, C, D\}$ to $\{1, 2, 3\}$ is equal to 4^3. 3 3 3 3

T F (2) The sequence $1, 3/2, 5/3, 7/6, 9/8, \ldots$ is increasing. 1 1.5 1.66 ≤1.5 X

T F (3) The sequence $-1, -2, -3, -4, \ldots$ is non-increasing.

T F (4) There are 4 anagrams of the word MAMA. $\frac{4!}{2!\,2!}$ $\cancel{\frac{4\,3\,2\,1}{4}}$ 6

There are 6 anagrams there are 4 anagrams?

T F (5) There are infinitely many Fibonacci numbers which are divisible by 3.

T F (6) The number of permutations of $\{1, 2, 3, 4, 5\}$ is smaller than 123. $5! = 5 \cdot 4$, $20 \cdot 3$, $60 \cdot 2$, 120

T F (7) The number of 3-permutations of $\{1, 2, 3, 4, 5, 6\}$ is equal to $\binom{6}{3}$. $\frac{n!}{n-k!} =$ $\frac{720}{6} = 120$ $\binom{6}{3} = \frac{n!}{k!(n-k!)}$ $\cancel{\frac{6!}{3!\,3!}}$

T F (8) The number of 3-subsets of $\{1, 2, 3, 4\}$ is equal to 4. $\binom{4}{3} = \frac{24}{3!\,1!}$ $\frac{24}{6} = 4$ ✓

T F (9) The number of permutations of $\{1, 2, \ldots, n\}$ which have n preceding $n - 1$ (not necessarily immediately) is equal to $n!/2$

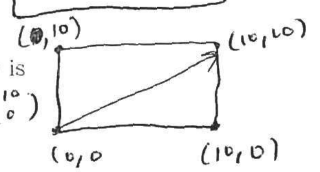
T F (10) For every $A, B \subset \{1, 2, \ldots, 12\}$ we have $|A \cap B| < |A \cup B|$.

T F (11) For all $n \geq 1$, we have

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}.$$

•(7, 3) •10,10 •(3,7)

T F (12) The number of grid walks from $(0, 0)$ to $(10, 10)$ going through $(3, 7)$ is equal to $\binom{10}{3}^2$. $\binom{10}{3}^2$



T F (13) The number of grid walks from $(0, 0)$ to $(10, 10)$ avoiding $(10, 0)$ and $(0, 10)$ is equal to $\frac{1}{2}\binom{20}{10}$. $\binom{20}{10} - 2\binom{10}{0}^2$ $\binom{20}{10} - \binom{10}{0}\binom{10}{0} - \binom{10}{0}\binom{10}{0}$

T F (14) The number of anagrams of MISSISSIPPI which begin with M is greater than the number of anagrams which begin with S.

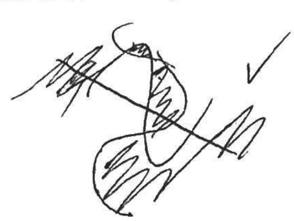
$\binom{10}{0} = \frac{2(10!)(10!)}{1}$

T F (15) The following parabolas are drawn in the plane:

$$y = x^2 - n^2x - n^3, \qquad n = 1, \ldots, 12.$$

$\frac{20!}{10!} - \frac{2(10!)(10!)}{1}$

Then the regions of the plane separated by these parabolas can be colored with two colors in such a way that no two adjacent regions have the same color.



✓