MIDTERM 1 (MATH 61, SPRING 2017)

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UCLA id: [redacted]

Math 61 Section: 1D

Date: 4/26/17

The rules:
You MUST simplify completely and BOX all answers with an **INK PEN**.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: those caught violating the rules get automatic 10% score deduction.

Points:

1	20
2	12
3	15
4	9
5	30

Total: 86 **(out of 100)**

Problem 1. (20 points)

Compute the probability that 4-subset A of $\{1, 2, \ldots, 10\}$ satisfies:

a) A has no odd numbers,

b) A has at least one number ≤ 3,

c) A contains 1 but not 7.

d) the smallest number in A is divisible by 3

$$\binom{10}{4} = \frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2}$$

$$\frac{\binom{5}{4}}{\binom{10}{4}}$$

a) $\dfrac{\binom{5}{4}}{\binom{10}{4}}$ $\boxed{\dfrac{\binom{5}{4}}{\binom{10}{4}}}$ ✓

all #s >3: $\binom{7}{4}$

b) $\dfrac{\binom{10}{4}-\binom{7}{4}}{\binom{10}{4}}$ $\boxed{\dfrac{\binom{10}{4}-\binom{7}{4}}{\binom{10}{4}}}$ ✓

$$\frac{\binom{10}{4}-\binom{7}{4}}{\binom{10}{4}}$$

$$\binom{8}{3}$$

c) $\dfrac{\binom{8}{3}}{\binom{10}{4}}$ $\boxed{\dfrac{\binom{8}{3}}{\binom{10}{4}}}$ ✓

3, 6, $\cancel{9}$

3: $\binom{7}{3}$

6: $\binom{4}{3}$

d) $\dfrac{\binom{7}{3}+\binom{4}{3}}{\binom{10}{4}}$ $\boxed{\dfrac{\binom{7}{3}+\binom{4}{3}}{\binom{10}{4}}}$ ✓

Problem 2. (20 points)

Let $X = \mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all non-negative integers. For each of the following functions $f : X \to X$ decide whether they are injective, surjective, bijective:

a) $f(x) = x + 1$

b) $f(x) = x^2 - 1$

c) $f(x) = 2x$

d) $f(x) = (x^2 + 2x)/(x + 2)$

a) injective $\boxed{\text{injective}}$

3

b) injective ?
or neither
$\boxed{\text{injective}}$

$0 \to -1$
$1 \to 0$
$2 \to 3$

2

c) injective
$\boxed{\text{injective}}$

$0 \to 0$
$1 \to 2$
$2 \to 4$

3

d) bijective
$\boxed{\text{bijective}}$

$0 \to 0$
$1 \to 1$

$\frac{x^2+2x}{x+2} = \frac{x\cancel{(x+2)}}{\cancel{x+2}} = x$

4

Problem 3. (15 points)

Let $a_n = 1111\cdots1$ (n ones). Suppose a_k is divisible by 97. Use induction to show that $a_{k\cdot n} = 0 \bmod 97$, for all $n \geq 1$.

15/15

a_3 $\quad K=3$

$a_3 \quad 111 \qquad a_4 = 1111$

$a_6 \quad 111{,}111$

$a_9 \quad 111{,}111{,}111$

Base: $n=1$

$a_{K\cdot 1} = a_K = 0 \bmod 97$ ✓

$a_6 = 10^K \cdot a_3 + a_3$

Step: Assume $a_{K\cdot n} = 0 \bmod 97$.

Prove $a_{K\cdot(n+1)} = 0 \bmod 97$

$a_{3+3} = a_3 +$

$a_{K(n+1)} = a_{\underbrace{K\cdot n}+K}$

$a_{K\cdot n} = 0 \bmod 97$ from inductive assumption

$a_{K\cdot n+K} = \underbrace{10^{\text{something}} \cdot a_K}_{\text{divisible by 97}} + \underbrace{a_{K\cdot n}}_{\text{divisible by 97}}$

So $a_{K\cdot n+K} = 0 \bmod 97$

$6+3$

$a_3 = 111 \qquad K=3 \quad n=2 \quad 9$

$a_6 = 111111$

$a_9 = 111111111$

1, 1, 2, 3, 5, 8, 13

4, 4, 6, 8, 12, 18, 28
1, 1, 2, 3, 5, 8, 13

 +3 +3 +4 +5 +7

Problem 4. (15 points)

Find closed formulas for the following sequences :

a) 4, 4, 6, 8, 12, 18, 28, ~~42~~ 44, 70, 112, . . .

b) $a_1 = 1$, $a_{n+1} = a_n \cdot \binom{n+1}{2}$ $\quad a_2 = a_1\binom{2}{2}$

c) $a_1 = 1$, $a_2 = 1$, $a_{n+1} = a_{n-1} - a_n$ for $n \geq 2$.

Note: you can express a_n in terms of Fibonacci numbers F_n.

$F_1\ F_2\ F_3\ F_4\ F_5$
1, 1, 2, 3, 5, 8 4, 4, 6, 8, 12, 18
$a_1 = 4$ 4, 4, 8, 12, 20, 32
$a_2 = 4$
$a_3 = 6$ −0 −0 −2 −4 −8 −16
$a_4 = 8$

a) $a_1 = 4$
$a_2 = 4$

4, 4, 6, 8, 12, 18, 28
1, 1, 2, 3, 5, 8, 13
4, 4, 8, 12, 20, 32,

$2^4 = 2 \cdot 2 \cdot 2 \cdot 2$
4 4
16

$a_4 = 4(2+1) - 2$
12 − 2 = 10

$a_n (a_{n-1} + a_{n-2}) - 2$

$a_1 = 4$
$a_2 = 4$

$a_n = 2F_n + 2$ $\boxed{a_n = 2F_n + 2}$ ✓

Difference between #s is $2F_n$

ok

$a_{n+1} = a_n\binom{n+1}{2}$

b)

$\frac{(n+1)!}{2!(n-1)!} = \frac{(n+1)(n)}{2}$

$a_{n+1} = a_n \frac{n(n+1)}{2}$

c) $a_n = (-1)^n ($

Boxes?

1, 1, 3 $\quad 1 \cdot \frac{2(3)}{2}$

$a_4 = 3 \cdot \frac{3(4)}{2}$
3 · 6

1, 1, 3, 18

1, 1, 2, 3, 5, 8
0 2 2 4 6 10 16
4, 4, 6, 8, 12, 18, 28, 44
1, 2, 3, 4, 5, 6

$a_4 = a_2 - a_3$
$a_3 = a_1 - a_2$

1, 2, 3, 4

1, 1, 0, 1, −1, 2, −3, 5, −8, 13
1 2 3 4 5 6 7 8 9 10
2+3 5+8

0, 1, 1, 2, 3, 5, 8

$a_5 = a_3 - a_4$

$a_1 = a_4 - a_5$

4, 4, 6, 8, 12
1, 2, 3, 4, 5

−3 −3

414
328
16

$a_3 = 6$
$F = 1$ 2+2
4+2 = 6

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

T **F** (1) The number of functions from $\{A, B, C, D\}$ to $\{1, 2, 3\}$ is equal to 4^3. 3^4

T **F** (2) The sequence $1, 3/2, 5/3, 7/6, 9/8, \ldots$ is increasing. $1, 1.5, 1.6, 1.1$ $5.0/3 = 1.6$ $7.0/6 = 1.1$

T F (3) The sequence $-1, -2, -3, -4, \ldots$ is non-increasing.

T **F** (4) There are 4 anagrams of the word MAMA. $\binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2} = 6$

T F (5) There are infinitely many Fibonacci numbers which are divisible by 3. $1, 1, 2, 3$

T F (6) The number of permutations of $\{1, 2, 3, 4, 5\}$ is smaller than 123. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ $20 \times 3 = 60$

T **F** (7) The number of 3-permutations of $\{1, 2, 3, 4, 5, 6\}$ is equal to $\binom{6}{3}$.

T F (8) The number of 3-subsets of $\{1, 2, 3, 4\}$ is equal to 4. $\binom{4}{3} = \frac{4 \cdot 3!}{3!}$

T F (9) The number of permutations of $\{1, 2, \ldots, n\}$ which have n preceding $n-1$ (not necessarily immediately) is equal to $n!/2$

T **F** (10) For every $A, B \subset \{1, 2, \ldots, 12\}$ we have $|A \cap B| < |A \cup B|$. $A = B$

T F (11) For all $n \geq 1$, we have

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}.$$

$1 + n(2n-1)$ $\frac{2n!}{2!(2n-2)!} = \frac{\cancel{2}n \cdot (2n-1)\cancel{(2n-2)!}}{\cancel{2}!}$

T F (12) The number of grid walks from $(0, 0)$ to $(10, 10)$ going through $(3, 7)$ is equal to $\binom{10}{3}^2$. $\binom{10}{3} \cdot \binom{10}{3}$

T **F** (13) The number of grid walks from $(0, 0)$ to $(10, 10)$ avoiding $(10, 0)$ and $(0, 10)$ is equal to $\frac{1}{2}\binom{20}{10}$.

T **F** (14) The number of anagrams of MISSISSIPPI which begin with M is greater than the number of anagrams which begin with S.

T F (15) The following parabolas are drawn in the plane:

$$y = x^2 - n^2 x - n^3, \qquad n = 1, \ldots, 12.$$

Then the regions of the plane separated by these parabolas can be colored with two colors in such a way that no two adjacent regions have the same color.

$x^2 + Cx + C$