Problem 1. (20 points)

Compute the number of permutations $(x_1, x_2, \ldots, x_n)$ of $\{1, 2, \ldots, 9\}$ such that:

a) $x_1 = 2$,

b) $x_1 \cdot x_2 \cdot x_3 = 6$,

c) $x_1 = x_2 = x_3 \bmod 7$,

d) $x_1 < x_2 < 5$.

ⓐ $x_1 = 2$

Then number of permutations

$\Rightarrow$ $8! \Rightarrow 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$

$\boxed{\text{ANS: } 40320 \text{ or } 8!}$

ⓑ $x_1 \cdot x_2 \cdot x_3 = 6$

So $x_1, x_2, x_3 \in \{1, 2, 3\}$

So $3! = 6$ ways of arranging x_1, x_2, x_3

and $6!$ ways of arranging the rest

$\boxed{\text{ANS}}$ $= \boxed{3! \cdot 6! = 4320}$ $= 6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

720
6
4320

ⓒ $x_1 = x_2 = x_3 \bmod 7$.

$1 = 1 \bmod 7$
$2 = 2 \bmod 7$
$3 = 3 \bmod 7$

$4 = 4 \bmod 7$
$5 = 5 \bmod 7$
$6 = 6 \bmod 7$

$7 = 0 \bmod 7$
$8 = 1 \bmod 7$
$9 = 2 \bmod 7$

clearly, it is not possible to have $x_1 = x_2 = x_3 \bmod 7$

$\boxed{\text{ANS}}$ $\boxed{0}$

ⓓ $x_1 < x_2 < 5$

There are 4 numbers < 5, and therefore $\cancel{4!}$ $(4! \cdot 7!)$ ways of arranging numbers such that $x_1 < x_2 < 5$. Out of these half are such that $x_1 < x_2$. So

ways to order $x_3 \ldots x_9 = 7!$

ANS = $\boxed{\frac{4!\,7!}{2}}$

Problem 2. (20 points)

Let $X = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ be the set of all integers. For each of these relations R, decide whether they are reflexive, symmetric or transitive (or neither).

a) xRy if and only if $|x| = |y|$.

b) xRy if and only if $x + 2y = 0 \bmod 3$.

c) xRy if and only if $x^2 + 2y^2 = 0 \bmod 3$.

d) xRy if and only if $x^3 + 122y^3 = 0 \bmod 3$.

(a) xRy iff $|x| = |y|$

→ ~~sy~~ Reflexive → $|x| = |x|$? Yes.

→ Symmetric? → $|x| = |y| \Rightarrow |y| \Rightarrow |x|$? → YES

→ Transitive → $|x| = |y|$ and $|y| = |z|$, then $|x| = |z|$? → YES

ANS. HENCE, reflexive, symmetric and transitive.

Equivalence

(b) Reflexive? → $x + 2y = 0 \bmod 3$

$x + 2x = 3x = 0 \bmod 3$ → YES ✓

Symmetric? → $x + 2y = 0 \bmod 3$

$2y + 3x = 0 \bmod 3$

$3x + 3y = 0 \bmod 3$

$x + 2y = 0 \bmod 3$

$2x + y = 0 \bmod 3$ → $\boxed{\text{YES}}$

Transitive? → $x + 2y = 0 \bmod 3$

$+\ y + 2z = 0 \bmod 3$

$x + 3y + 2z = 0 \bmod 3$ → (1)

but $3y = 0 \bmod 3$ → (2)

Subtracting (1) from (2), we get $x + 2z = 0 \bmod 3$ → YES

$\boxed{\text{ANS}}$ → Reflexive, symmetric and transitive.

Equivalence

CONTINUED →

(c) xRy iff $x^2 + 2y^2 = 0 \bmod 3$

(a) Reflexive?

$x^2 + 2x^2 = 0 \bmod 3$

$3x^2 = 0 \bmod 3 \rightarrow$ YES, Reflexive

(b) $x^2 + 2y^2 = 0 \bmod 3 \rightarrow (1)$

$3x^2 + 3y^2 = 0 \bmod 3 \rightarrow (2)$

(2)-(1) $2x^2 + y^2 = 0 \bmod 3 \rightarrow$ Hence symmetric.

(c) Transitive?

xRy $x^2 + 2y^2 = 0 \bmod 3$

yRz $y^2 + 2z^2 = 0 \bmod 3$

$x^2 + 3y^2 + 2z^2 = 0 \bmod 3 \rightarrow (3)$

But $3y^2 = 0 \bmod 3 \rightarrow (4)$

Subtracting (4) from (3), gives

$x^2 + 2z^2 = 0 \bmod 3 \rightarrow$ Hence transitive

$\boxed{\text{ANS: Reflexive, Symmetric and Transitive}}$

(d) xRy iff $x^3 + 122y^3 = 0 \bmod 3$

(a) Reflexive?

$x^3 + 122x^3 = 0 \bmod 3$?

$123x^3 = 0 \bmod 3 \rightarrow$ ✓ Reflexive ✓

(b) Symmetric?

$123x^3 + 123y^3 = 0 \bmod 3$

$x^3 + 122y^3 = 0 \bmod 3$

$122x^3 + y^3 = 0 \bmod 3 \rightarrow$ Symmetric ✓

(c) transitive?

$x^3 + 122y^3 = 0 \bmod 3$

$y^3 + 122z^3 = 0 \bmod 3$

$x^3 + 123y^3 + 122z^3 = 0 \bmod 3$

But $123y^3 = 0 \bmod$ therefore

$x^3 + 122z^3 = 0 \bmod$

$\rightarrow$ transitive ✓

$\boxed{\text{ANS: REFLEXIVE, SYMMETRIC, TRANSITIVE}}$

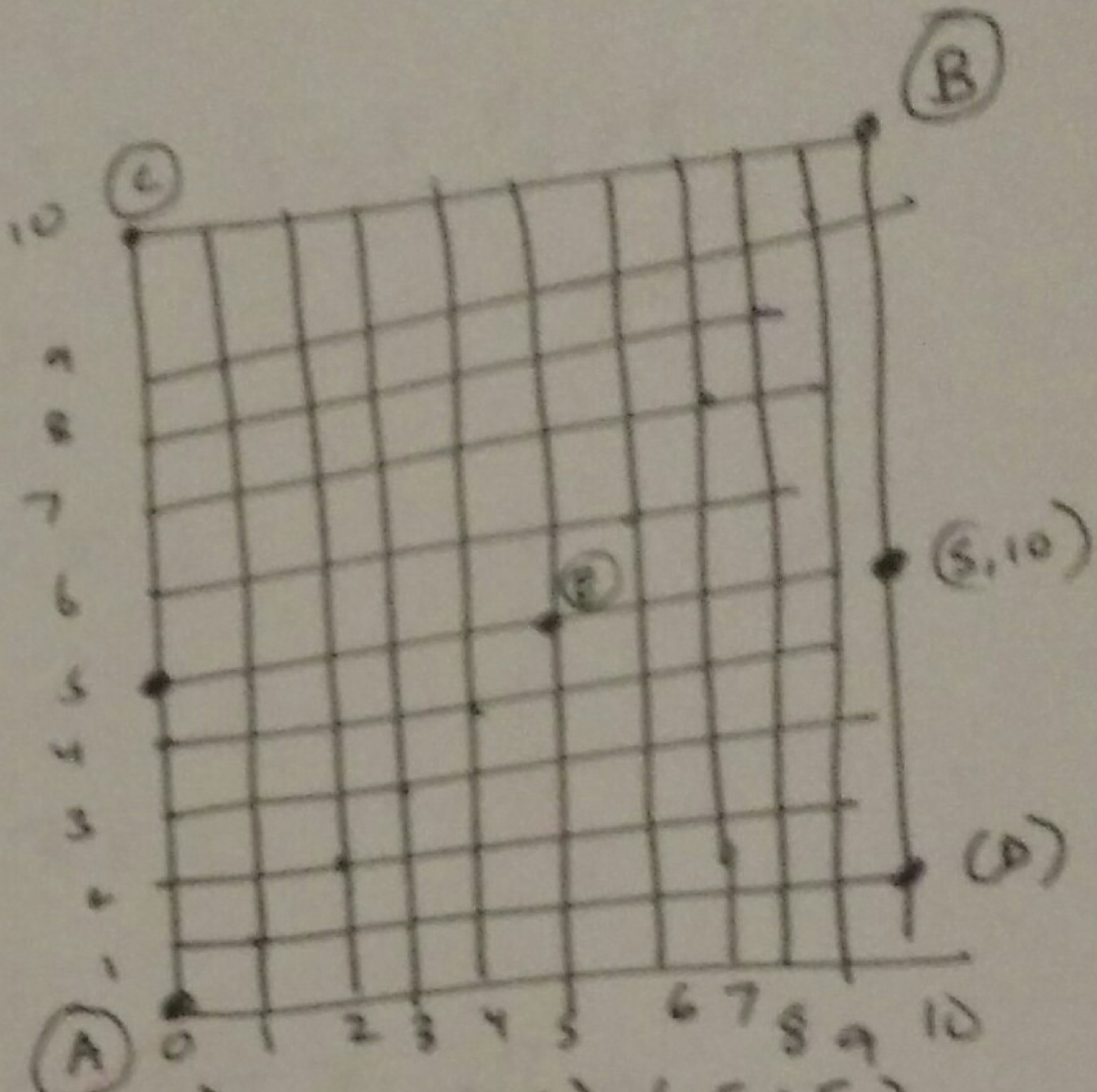
Problem 3. (15 points)

Let $A = (0,0)$, $B = (10,10)$. Find the number of (shortest) grid walks γ from A to B, such that:

a) γ never visits points $\overset{C}{(0,10)}$, $\overset{D}{(10,1)}$, $\overset{E}{(5,5)}$.

b) γ visits all points $(1,1), (2,2), (3,3), \ldots, (9,9)$.

c) γ visits points $(5,0)$ and $(5,10)$, but not $(5,5)$.



(a) ~~never~~

\# from A to B

− # from A → C → B

− # from A → D → B

− # from A → E → B

+ # from A → C → D → E → B = 0 (since only shortest)

$$\Rightarrow \text{~~ANS~~} = \binom{20}{10} - \binom{10}{10}\binom{10}{10} - \binom{10+1}{1}\binom{9+0}{0} - \binom{5+5}{5}\binom{5+5}{5}$$

$$\boxed{\text{ANS} = \binom{20}{10} - 1 - \binom{11}{1}\binom{9}{0} - \binom{10}{5}\binom{10}{5} = \binom{20}{10} - \binom{10}{5}^2 - 12}$$

(b) Visits all points

$$\Rightarrow \cancel{\binom{20}{10} - \binom{2}{1} - \binom{4}{2} - \binom{6}{3}}$$

$$\Rightarrow \binom{2}{1} + \binom{2}{1} + \binom{2}{1} \cdots$$

$$\boxed{\text{ANS} \Rightarrow 9\binom{2}{1} \Rightarrow 18}$$

(c) Visits (5,0) and (5,10) but not (5,5)

Must visit (5,0) first and then (5,10) for shortest walks.

So for (5,0) $\binom{5+0}{5}$. for $\binom{(5,0)}{(5,10)} \Rightarrow \binom{10+0}{0} = 1$

$\boxed{\text{ANS} = 0}$

Since all paths in this way must pass through (5,5)

Problem 4. (15 points)

Recall the Fibonacci sequence: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, etc.

Prove that $F_n \leq 2^{n-1}$.

$F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$.

BASE CASE : $F_1 \leq 2^{1-1}$?

TRUE, as $F_1 = 1 \leq 1$.

$F_2 \leq 2^{2-1}$?

TRUE, as $F_2 = 1 \leq 2$.

INDUCTION HYPOTHESIS : $F_n \leq 2^{n-1}$ is true for all n.

To prove this is true for $n+1$?

~~PROOF~~ STEP To prove $F_{n+1} \leq 2^{(n+1)-1}$

or $F_{n+1} \leq 2^n$

$F_{n+1} = F_n + F_{n-1}$

But $F_n \leq 2^{n-1}$ and $F_{n-1} \leq 2^{n-2}$ } from hypothesis

So $F_{n+1} = F_n + F_{n-1} \leq 2^{n-1} + 2^{n-2}$

$\leq 2^{n-2}(2+1)$

$\leq 3(2^{n-2})$

However $2^{n-2} \leq (4 \cdot 2^{n-2} = 2^n)$ clearly.

Therefore $F_{n+1} \leq 3(2^{n-2}) \leq (4 \cdot 2^{n-2} = 2^n)$

Thus proved

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

T F (1) The number of functions from $\{A, B, C, D\}$ to $\{1, 2, 3\}$ is equal to 4^3.

T F (2) The sequence $10, 21, 32, 43, \ldots$ is increasing.

T F (3) The sequence $2/1$, $3/2$, $4/3$, $5/4$ is non-increasing.

T F (4) There are 20 anagrams of the word *BUBUB*.

T F (5) There are more anagrams of the words *AAAACCC* which begin with *A* than with *C*. $A \to \frac{6!}{3!3!} = \frac{6\cdot5\cdot4}{6} = 20$ $C \to \frac{6!}{4!2!} = \frac{6\cdot5}{2} = 15$

T F (6) There are infinitely many Fibonacci numbers $= 1 \bmod 3$.

T F (7) There are infinitely many binomial coefficients $\binom{n}{k} = 1 \bmod 17$.

T F (8) Each of the 14 students wrote on a paper 10 distinct numbers, from the set $\{1, 2, \ldots, 100\}$. Then there are two students who have at least 2 numbers in common on their lists. Total 140 numbers were on the 100 distinct numbers

T F (9) The probability that a random 10-subset of $\{1, 2, \ldots, 19\}$ contains 10 is equal to 1/2. contains $\binom{10}{9}$ $\binom{18}{9}$ $\binom{19}{10}$

T F (10) For every two subsets $A, B \subset U$, we must have $|A \setminus B| = |B \setminus A|$. A − B = B − A

T F (11) For every two subsets $A, B \subset U$, we must have $|A \cup B| \geq |\overline{B}|$

T F (12) Every surjection that is also a bijection must be also an injection. surjection

T F (13) Every surjection that is also an injection must be also a bijection.

T F (14) Let A be the set of 3-subsets of $[9] = \{1, 2, \ldots, 9\}$. Similarly, let B be the set of 6-subsets of $[9]$. Consider a map $f : A \to [9] \setminus A$. Then f is a bijection from A to B.

T F (15) The pigeon hole principle was proved in class by induction. contradiction.

$\binom{9}{3}$

Surjection → surjection can have multiple preimages. But if bijection, only one pre image.

A 2 3 4

B 2

(A∪B)

A∪B is always bigger than A and B