Second Midterm Examination, Version 1

Math. 61, Winter Quarter, 2009

Instructor: H. Hida

Friday, February 27, 2008, 1:00 p.m.-1.50 p.m.

Print your name: BOYKO BORYS

last middle initial first

Student ID number: [redacted] - [redacted]

Sign in full name: [redacted]

1.	65	/80
2.	60	/75
3.	43	/45
Total	168	/200

Note:

(1) Keep your desktop clean. Put your textbooks and notebooks in your bag and keep them closed.

(2) Do not use any scrap papers. You may use the back side of the exam papers for computation.

(3) You may use calculators. Save all your computations for partial credits.

(4) There are 3 problems in this book. This book contains 4 pages including this page.

1. Compute the following numbers, explain how you get the answer and write your answer in the following boxes as indicated:

a. ~~52~~ 592	b. −3645	c. −2240	d. 720

a. How many integers between 1 and 10,000 have the sum of the digits equal to 15?

$0 \le x_i \le 9$

$x_1 + x_2 + x_3 + x_4 = 15$

all are ≤ 9 and ≥ 0

nonnegative solutions

$x_1 - 10 + x_2 + x_3 + x_4 = 5$

$y_1 + x_2 + x_3 + x_4 = 5$

$\binom{15+3-1}{3-1} - 4\binom{5+3-1}{3-1}$

~~$\boxed{52}$~~

$C(8,3) = 56$

$C(18,3) = 816$

$816 - 4(56) = 592$

b. a_6 for the sequence of numbers a_n satisfying $a_0 = 1$ $a_1 = 0$ and $a_n = 6a_{n-1} - 9a_{n-2}$.

$a_0 = 1$

$a_1 = 0$

$a_2 = -9$

$a_3 = -54$

$a_4 = -243$

$a_5 = -972$

$a_6 = \boxed{-3645}$

c. The coefficient of $x^2y^3z^3$ in the expansion of $(2x + y - z)^8$.

$$\binom{8}{2}\binom{6}{3}(2)^2(-1)^3 = \boxed{-2240}$$

d. There are 9 blue balls, 5 red balls and 8 green balls in a bag. In how many ways we can draw 8 blue balls, 2 red balls and 7 green balls if the balls are considered distinct.

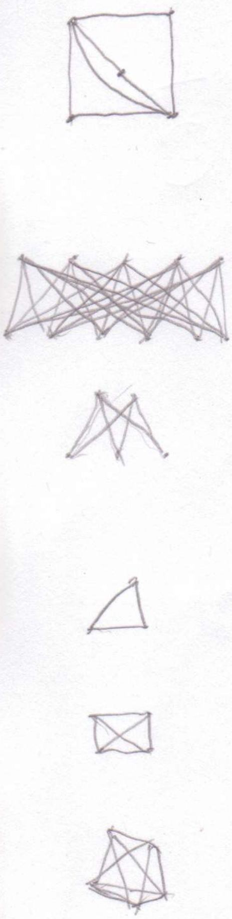
~~$\frac{22!}{8!\,2!\,7!}$~~

$$\binom{9}{8}\cdot\binom{5}{2}\cdot\binom{8}{7} = \boxed{720}$$

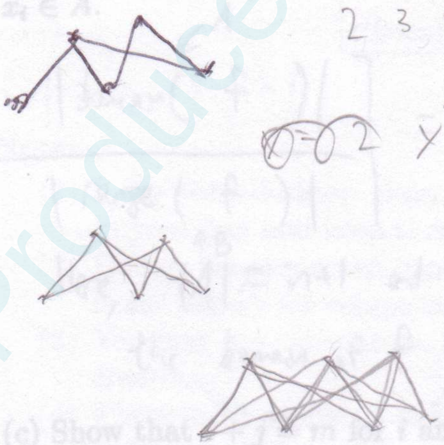
$\binom{6}{1} + \binom{6}{3} + \binom{6}{5}$

$\sum_{k=1}^{3} C(6, 2k-1)$ $\sum_{k=1}^{2} C(4, 2k-1)$ $\binom{4}{2} + \binom{4}{3}$

2. Label the following statement as being true or false.



Statements	Label
If n is an even positive integer, $\sum_{k=1}^{n/2} C(n, 2k-1) = 2^{n-1}$.	True
If a graph has a Hamiltonian cycle, removing one vertex (and edges coming out of the vertex) does not disconnect the ~~garph~~ graph. -1	True
There is a simple graph of 5 vertices out of which 3 vertex have degree 3 and 2 vertices have degree 2. not even sum of degrees	False
$a_n = 2(1+\sqrt{-1})^n + 2(1-\sqrt{-1})^n$ is a solution of $a_n = 2a_{n-1} - 2a_{n-2}$.	False ✗ True
There is a simple graph with 5 vertices having two vertices of degree 4 and at least one vertex of degree 1.	False
The complete bipartite graph $K_{5,6}$ has a simple 3 cycle.	False
Every complete graph K_n $(n \geq 3)$ has an Euler cycle.	True ✗ False
A complete bipartite graph $K_{n,n-1}$ $(n \geq 3)$ has a Hamiltonian cycle.	False
A complete bipartite graph $K_{n,n-1}$ $(n \geq 3)$ has a Euler cycle.	False
The sum of degrees of all vertices of a graph can be odd or even.	False
If a graph has an Euler cycle, it has a Hamiltonian cycle.	False
If all the vertices of a graph have even degree, the graph has an Euler cycle.	True ✗ False
A recurrence relation can have infinitely many distinct solutions.	True
If a graph H has a simple cycle of length k, it always has a simple cycle of length less than k.	False
For any three positive integers m, n, k, $m!n!k!$ is a factor of $(m+n+k)!$.	True



2 3 5

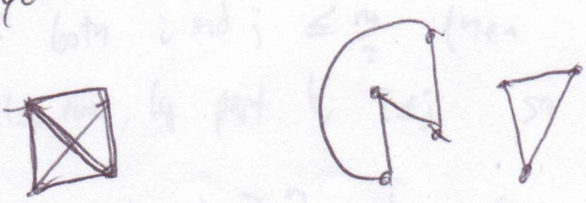
$2e =$ sum degrees

$x^2 - 2x - 2 = 0$

$\frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$

$1 \pm i$

Euler cycle – visits each edge once



−3

$|\text{dom } f| = n+1$

domain of f: $\overset{n+2}{X} - \overset{1}{\{m\}}$ is size $n+2-1 = n+1$

3. Let X be an $(n+2)$–element subset of $\{1, 2, 3, \ldots, 2n+1\}$ and m be the greatest element in X. Define a function

$$f : X - \{m\} \to \{1, 2, 3, \ldots, 2n+1\} \text{ by } f(k) = \begin{cases} k & \text{if } k \le \frac{m}{2}, \\ m-k & \text{if } k > \frac{m}{2}. \end{cases}$$

(a) Show that the range of f is contained in $Y = \{1, 2, \ldots, n\}$.

$m \le 2n+1$

$\frac{m}{2} \le \frac{2n+1}{2}$ $\quad \frac{m}{2} \le n + \frac{1}{2}$ $\quad \frac{m}{2}$ is an integer, so

$\frac{m}{2} \le \lfloor n + \frac{1}{2} \rfloor = n$

This means that if $k \le \frac{m}{2}$ then $k \le n$

and if $k > \frac{m}{2}$ then $(m-k) < n$

$\therefore$ the range is $\{1, 2, \ldots, n\}$

the number of elements present in the range?

−2

Why ≥ 1?

(b) Use the following pigeon hole principle to show that $f(i) = f(j)$ for some $i \ne j$.

Pigeon Hole Principle: *If $f : A \to B$ is a function for finite sets A and B and $n = |A| > |B| = m$, then for the smallest integer ℓ greater than or equal to $\frac{n}{m}$, $f(x_1) = f(x_2) = \cdots = f(x_\ell)$ for some distinct $x_i \in A$.*

$$\left\lceil \frac{|\overset{A}{\text{domain}(f)}|}{|\underset{B}{\text{range}(f)}|} \right\rceil = \left\lceil \frac{|X - \{m\}|}{n} \right\rceil = \left\lceil \frac{n+1}{n} \right\rceil = 2$$

Here $|A| = n+1$ and $|B| = n$, $|A| > |B|$, so $\exists\, x_1, x_2$ in the domain of f such that $x_1 \ne x_2$ and $f(x_1) = f(x_2)$

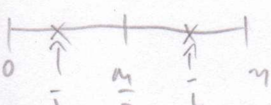
$f(i) = f(j)$ for some $i \ne j$ (replace x_1 with i and x_2 with j)

(c) Show that $i + j = m$ for i and j in (b).

If both i and j $> \frac{m}{2}$ then $i = j$

If both i and j $\le \frac{m}{2}$ then $i = j$

However, by part b, $i \ne j$ so one has to be $\le \frac{m}{2}$ and the other has to be $> \frac{m}{2}$



so, assume $i > \frac{m}{2}$ and $m - i \le \frac{m}{2}$ then $\Rightarrow f(i) = f(j) = m - i$

$j = m - i$ and $j + i = m \Rightarrow i + j = m$ ✓