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Math 61
Midterm I
Feb 8, 2013

Name:
Please put your **last** name first and print clearly

Signature:.

TA section you are attending:
(Tues or Thurs, name of TA, section letter)

1. 20 2. 20

3. 20 4. 20

5. 20 6. 20

7. 20

You must put all your answers in the spaces provided on the page of the problem . **Please do not use the spaces on this page.** You must show a method of solution to obtain credit for a problem. You need not simplify your answers and you can leave your answers in terms of $C(n,r)=\binom{n}{r}$ or $P(n,r)$. NO CALCULATORS!

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1. Let $X = \{0, 1, 2, 3, 4\}$ and let $f(x) = ((x^2 - x) \bmod 5)$. For $x, y \in X$ define an equivalence relation R by

$$xRy \text{ if and only if } f(x) = f(y)$$

What are the distinct equivalence class? How many distinct equivalence classes are there? (You do not have to prove R is an equivalence relation. $n \bmod 5$ is the remainder of n when divided by 5)

How many distinct equivalence classes : 3

List them : $[0] = [1] = \{\cancel{(0,0), (0,1), (1,0), (1,1)}\}$ 0,1

$[2] = [4] = \{\cancel{(2,2), (2,4), (4,2), (4,4)}\}$

$[3] = \{(3,3)\}$ 2,4

$f(0) = 0$ $\quad [0] = [1] = \{(0,0), (0,1)\ (1,0)\ (1,1)\}$

$f(1) = 0$ $\quad [2] = [4] = \{(2,4)\ (4,2)\ (2,2).\ (4,4)$

$f(2) = 2$ $\quad [3] = \{(3,3)\}$

$f(3) = 1$

$f(4) = 2$

2. Show that for $n \in \mathbf{Z}$, $n \geq 1$

$$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \ldots + \frac{1}{(n)(n+1)} = \frac{n}{n+1}$$

Base: for $n=1$, $\frac{1}{1(1+1)} = \frac{1}{1+1}$

$\frac{1}{2} = \frac{1}{2}$ (base case works)

Inductive Hypothesis: Let us assume that $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \ldots + \frac{1}{(n)(n+1)} = \frac{n}{n+1}$

Then, $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)}$$

$$= \frac{n^2+2n+1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)}$$

$$= \frac{n+1}{n+2}$$

$\therefore$ for $n \in \mathbb{Z}, n \geq 1$, $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

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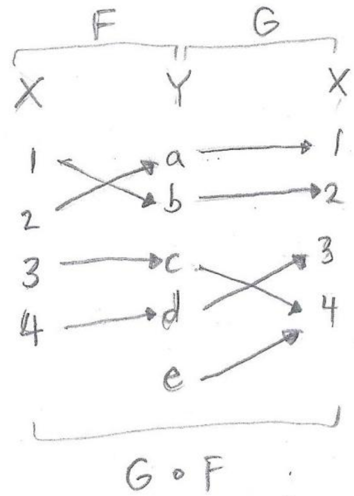
3. Suppose $F : X \to Y$ and $G : Y \to X$. Suppose $G \circ F : X \to X$ is one to one and onto. True or false: F is always onto. Either provide a proof or give a counterexample. If you give a counterexample, give the arrow diagram of your functions. Explain.



$\boxed{\text{False}}$

If Y contains more elements than X ($|Y| > |X|$), then F cannot be onto, yet $G \circ F$ can be, as long as each element of Y that is "hit" by X is mapped onto a unique element of X. In the diagram, F is not onto because the element "e" is not "hit", but $G \circ F$ is onto because each element in X is "hit" by an element of Y that was "hit" by an element of X.

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4. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $Y = \{a, b\}$ How many functions are there from X to Y which are onto? Hint: First determine the total number of functions.

Answer: $2^9 - 2$ functions

Total number of functions : 2^9 (each of 9 elements has 2 choices)

Functions that are NOT onto all go to one element : 2 (all "a" or all "b")

Answer : $2^9 - 2$ or 510

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5. A committee of six people, A, B, C, D, E, F has to select a president, a secretary and a treasurer. How many ways can this be done if either D is president or D is not an officer?

√

Answer: 80 ways

Ways D is pres. = $\underbrace{5}_{\text{ways to choose secretary}} \times \underbrace{4}_{\text{ways to choose treasurer}} = 20$

Ways D is not officer: $\underbrace{5}_{\text{ways to choose president}} \times \underbrace{4}_{\text{ways to choose secretary}} \times \underbrace{3}_{\text{ways to choose treasurer}} = 60$

Ways D is pres. and not officer = 0 (it's impossible)

Ways D is pres. or not officer = 20 + 60 − 0 = 80

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6. The domain of definition in this problem is the reals $\mathbf{R}$. True or false:

$$\forall y\, \exists x\ ((x < y) \to (x^2 < y^2)).$$

Either prove this or give a counterexample. Explain in complete sentences.

~~False for $y=0$, there does NOT exist an x less than y such that $x^2 < y^2$ because $x^2 < 0$ is an impossible statement.~~

~~Equivalent: $\exists y\, \forall x\, (\neg((x<y) \to (x^2<y^2)))$~~

~~There exists y ($y=0$) such that for all x, $(x<y) \to (x^2<y^2)$ is not true~~

True, for all $y \in \mathbb{R}$, there exists x such that $((x<y) \to (x^2<y^2))$ is true, because there exists an $x \geq y$ in the set of all real numbers. If $x \geq y$, $((x<y) \to (x^2<y^2))$ would be vacuously true. Thus, $\forall y \exists x\, ((x<y) \to (x^2<y^2))$ is a true statement.

7. Suppose X has n elements and Y has three elements. How many **onto** functions are there from X to Y? This problem is difficult. Advice: only try this after you have done the rest of the problems.

Answer: $3^n - 3(2^n-1)$ functions for $n \geq 1$ (0 if $n=0$)

Total functions $= 3^n$ (each of n has 3 choices)

How many functions NOT onto? Must go only to 1 or 2 elements of Y

1 element only: 3 ways

2 elements: 1st and 2nd: 2^n-2 (each element either goes to 1st or to 2nd but don't count "all in 1")

2nd and 3rd: 2^n-2 (same as above)

1st and 3rd: 2^n-2 (same as above)

Total onto functions $= 3^n - 3(2^n-2) - 3$

$= 3^n - 3(2^n-1)$