Math 61 MT2

JEREMY TSAI

TOTAL POINTS

55 / 58

QUESTION 1

15 pts

1.1 5 / 5

✓ **- 0 pts** Correct

- 2 pts missing uniqueness

- 1 pts Missing quantifier "for all x"

- 2 pts No mention of relation

- 5 pts Blank or entirely incorrect

1.2 5 / 5

✓ **- 0 pts** Correct

- 1 pts One wrong quantifier

- 3 pts Injection wrong

- 2 pts Surjection wrong

- 5 pts Blank or completely incorrect

1.3 5 / 5

✓ **- 0 pts** Correct

- 1 pts Minor error

- 3 pts "Since g □ f = h □ f and f □ g = f □ h, then g = h" with no justification.

- 4 pts Some work, did not get far

- 5 pts Blank/entirely incorrect

QUESTION 2

10 pts

2.1 5 / 5

✓ **- 0 pts** Correct

- 1 pts error in scope

- 1 pts one error in the conditions

- 1 pts one error in the conditions

- 1 pts one error in conclusion

- 4 pts mostly wrong, but had some resemblance to theorem

- 4 pts different theorem/proposition about gcd or divisibility

- 5 pts no relevant work

2.2 5 / 5

✓ **+ 5 pts** Correct

+ 1 pts express A=B mod n as n|A-B

+ 1 pts factor ac-ab = c(a-b)

+ 1 pts mention Euclid's lemma

+ 1 pts Alternative approach to Euclid's lemma: Bezout's identity

+ 0 pts Incorrect

QUESTION 3

15 pts

3.1 2 / 5

- 0 pts Correct (51!)

- 0.5 pts Didn't allow for empty lines (e.g. 49 * 50!)

- 2 pts Chose a global order (e.g. 2^50 * 50!)

- 2.5 pts Didn't choose an order (e.g. 2^50)

- 2.5 pts Only made one line (e.g. ∑50!/(50-n)!)

- 2.5 pts 2*50! (or similar) with some explanation

✓ **- 3 pts ∑n!(50-n)! or similar**

- 3 pts Treated customers as indistinguishable (e.g. 51)

- 4 pts 50 choose 2

- 4 pts 50 permute 2

- 5 pts Nothing written / no significant progress

3.2 5 / 5

✓ **- 0 pts** Correct

- 1 pts Missing or incorrect step

- 1 pts Arithmetic mistake

- 1 pts Went past stopping condition

- 2.5 pts Wrong answer

- **2.5 pts** Didn't use Euclidean Algorithm

3.3 5 / 5

✓ **- 0 pts** Correct
- **1 pts** Calculation Mistake
- **1 pts** Flipped x & y
- **1 pts** Sign error
- **5 pts** No significant progress

QUESTION 4

18 pts

4.1 2 / 2

✓ **- 0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.2 2 / 2

✓ **- 0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.3 2 / 2

✓ **- 0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.4 2 / 2

✓ **- 0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.5 2 / 2

✓ **- 0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.6 2 / 2

✓ **- 0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.7 2 / 2

✓ **- 0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.8 2 / 2

✓ **- 0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.9 2 / 2

✓ **- 0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

Math 61, Winter 2020
Introduction to Discrete Structures
Midterm Exam 2

February 21, 2020

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient work/explanations, unless the problem explicitly states that no work is necessary. You can not use any notes, books, or electronic devices of any kind during the exam. If you have a question about any particular problem, please raise your hand. At the completion of the exam, please hand in the exam booklet. If you have any questions about the grading of the exam, please consult the midterm regrade policy in the course syllabus.

Name: Jeremy Tsai

UID: 105335484

I certify that the work appearing on this exam is completely my own.

Signature: ______________________

Question:	1	2	3	4	Total
Points:	15	10	15	18	58
Score:					

1. Do the following:

   (a) Suppose $f \subseteq X \times Y$ is a relation. State in precise mathematical language (for instance, using quantifiers) what it means for f to be a *function* $f : X \to Y$. (5)

   (b) Suppose $f : X \to Y$ is a function. State separately (again, in precise mathematical language) what it means for f to be (i) an *injection* and (ii) a *surjection*. (5)

   (c) Suppose $f : X \to Y$ is a function. Recall that an *inverse* of f is a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Suppose $g, h : Y \to X$ are both inverses of f. Prove that $g = h$. Justify all steps. (5)

a) $\forall x \in X\ \exists y \in Y\ ((x,y) \in f)$

$\forall y_1 \in Y\ \forall y_2 \in Y\ \forall x \in X\ ((x,y_1) \in f \land (x,y_2) \in f \to (y_1 = y_2))$.

b) $\forall x_1 \in X\ \forall x_2 \in X\ (f(x_1) = f(x_2) \to x_1 = x_2)$ Injection

$\forall y \in Y\ \exists x \in X\ (f(x) = y)$ Surjection

c) By definition of inverse $g \circ f = id_X$, $f \circ g = id_Y$, $h \circ f = id_X$, $f \circ h = id_Y$

$g = g \circ id_Y = g \circ f \circ h = id_X \circ h = h$. QED

- $g = g \circ id_Y$: by property of identity function
- $= g \circ f \circ h$: substituting by def of inverse
- $= id_X \circ h$: associativity of composition
- $= h$: property of identity function.

2. Do the following:
   (a) State *Euclid's Lemma.* (5)
   (b) Let $a, b, c \in \mathbb{Z}$ and $n \geq 1$ be arbitrary. Suppose $ac \equiv bc \pmod{n}$ and $\gcd(c, n) = 1$. Prove that $a \equiv b \pmod{n}$. (5)

a) If $(a,b) \neq (0,0)$ and $\gcd(a,b) = 1$.
$a | bc \Rightarrow a | c$.

b) $ac \equiv bc \pmod{n} \rightarrow n | ac-bc = n | c(a-b)$ (definition)

by Euclid's Lemma, $\gcd(c,n) = 1$, $(c,n) \neq (0,0)$, so
$n | a-b$. So $a \equiv b \pmod{n}$ by definition. QED

3. Do the following:

   (a) Suppose at a super market there are 2 check-out lines (i.e., queues) and 50 customers. How many ways can these 50 customers get into 2 lines? Here the lines and customers are distinguishable and their order in each line matters. Also, lines are allowed to be empty. For full points, your final answer cannot contain a summation (i.e., no "$\sum$" or "$+\cdots+$".) (5)

   (b) Use the Euclidean Algorithm to compute $\gcd(77, 53)$. (5)

   (c) Use part (b) to find integers $x, y \in \mathbb{Z}$ such that $77x + 53y = \gcd(77, 53)$. (5)

a) $\boxed{\sum_{k=0}^{50} k!(50-k)!}$

$\binom{50}{k} = \frac{50!}{(k!)(50-k)!} \Rightarrow \frac{50!}{\binom{50}{k}} = k!(50-k)!$

$\frac{50!}{2^{50}}$

0 2 → 2
1 1 → 2
2 0 → 2
6

b) $\gcd(77,53) = \gcd(53,24) = \gcd(24,5)$
$= \gcd(5,4) = \gcd(4,1) = \boxed{1}$

$77 = 53*1+24$
$53 = 24*2+5$
$24 = 5*4+4$
$5 = 4*1+1$
$4 = 1*4+0$

c) $1 = 5+4(-1)$
$= 5+(24+5(-4))(-1) = 5(5)+24(-1)$
$= (53+24(-2))5+24(-1)$
$= 53(5)+24(-11) = 53(5)+(77+53(-1))(-11)$
$= 53*16+77(-11)$

$\boxed{x=-11, y=16}$

4. For each of the following statements, indicate whether they are True or False. **A blank answer will receive 1 point.** [Recall: *True* means the same thing as "always true" and *False* means the same thing as "there exists a counterexample".] No work is necessary for this problem.

(a) True Suppose a group of 100 people want to form a committee of 10 people. There are $\binom{100}{90}$ ways to do this. (2)

(b) False Suppose $a, b, c \in \mathbb{Z}$. If $a|bc$, then either $a|b$ or $a|c$. (2)

(c) True. Suppose $f, g, h, i : X \to X$ are functions such that $f \circ g \circ h \circ i : X \to X$ is surjective. Then $f \circ g : X \to X$ is surjective. (2)

(d) False. Suppose $a, b \in \mathbb{Z}$ and $(a, b) \neq (0, 0)$. If $\gcd(a, b) = 1$, then $\gcd(a + b, a - b) = 1$. (2)

(e) False. Suppose X, Y are finite sets with $|X| = n$ and $|Y| = k$. Then the total number of functions $X \to Y$ is n^k. (2)

(f) True. Suppose n is an odd positive integer. Then (2)

$$\left|\{(x_1, x_2) \in \mathbb{N}^2 : x_1 + x_2 = n\}\right| \% 2 = 0.$$

(g) True Suppose $x, y \in \mathbb{Z}$ are arbitrary, p is a prime number and $n \geq 1$ is arbitrary. Then $(x + y)^{p^n} \equiv x^{p^n} + y^{p^n} \pmod{p}$. (2)

(h) False. Suppose X, Y, Z are *finite* sets and $f : X \to Y$ and $g : Y \to Z$ are functions. If f is injective and g is surjective, then $|X| \leq |Z|$. (2)

(i) True. Suppose X is a finite set. Then $|X| < |\mathcal{P}(X)|$. (2)