Math 61 MT2

JUSTIN CHAO

TOTAL POINTS

45 / 58

QUESTION 1

15 pts

1.1 5 / 5

- ✓ **- 0 pts** **Correct**
- **- 2 pts** missing uniqueness
- **- 1 pts** Missing quantifier "for all x"
- **- 2 pts** No mention of relation
- **- 5 pts** Blank or entirely incorrect

1.2 2 / 5

- **- 0 pts** Correct
- **- 1 pts** One wrong quantifier
- ✓ **- 3 pts** **Injection wrong**
- **- 2 pts** Surjection wrong
- **- 5 pts** Blank or completely incorrect

1.3 5 / 5

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Minor error
- **- 3 pts** "Since g ▯ f = h ▯ f and f ▯ g = f ▯ h, then g = h" with no justification.
- **- 4 pts** Some work, did not get far
- **- 5 pts** Blank/entirely incorrect

QUESTION 2

10 pts

2.1 5 / 5

- ✓ **- 0 pts** **Correct**
- **- 1 pts** error in scope
- **- 1 pts** one error in the conditions
- **- 1 pts** one error in the conditions
- **- 1 pts** one error in conclusion
- **- 4 pts** mostly wrong, but had some resemblance to theorem
- **- 4 pts** different theorem/proposition about gcd or divisibility
- **- 5 pts** no relevant work

2.2 1 / 5

- **+ 5 pts** Correct
- **+ 1 pts** express A=B mod n as n|A-B
- **+ 1 pts** factor ac-ab = c(a-b)
- ✓ **+ 1 pts** **mention Euclid's lemma**
- **+ 1 pts** Alternative approach to Euclid's lemma: Bezout's identity
- **+ 0 pts** Incorrect

QUESTION 3

15 pts

3.1 2 / 5

- **- 0 pts** Correct (51!)
- **- 0.5 pts** Didn't allow for empty lines (e.g. 49 * 50!)
- **- 2 pts** Chose a global order (e.g. 2^50 * 50!)
- **- 2.5 pts** Didn't choose an order (e.g. 2^50)
- **- 2.5 pts** Only made one line (e.g. ∑50!/(50-n)!)
- **- 2.5 pts** 2*50! (or similar) with some explanation
- ✓ **- 3 pts** **∑n!(50-n)! or similar**
- **- 3 pts** Treated customers as indistinguishable (e.g. 51)
- **- 4 pts** 50 choose 2
- **- 4 pts** 50 permute 2
- **- 5 pts** Nothing written / no significant progress

3.2 5 / 5

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Missing or incorrect step
- **- 1 pts** Arithmetic mistake
- **- 1 pts** Went past stopping condition
- **- 2.5 pts** Wrong answer

- **2.5 pts** Didn't use Euclidean Algorithm

3.3 5 / 5

✓ - **0 pts** Correct
- **1 pts** Calculation Mistake
- **1 pts** Flipped x & y
- **1 pts** Sign error
- **5 pts** No significant progress

QUESTION 4

18 pts

4.1 2 / 2

✓ - **0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.2 2 / 2

✓ - **0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.3 2 / 2

✓ - **0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.4 2 / 2

✓ - **0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.5 1 / 2

- **0 pts** Correct
- **2 pts** Incorrect

✓ - **1 pts** Blank

4.6 1 / 2

- **0 pts** Correct
- **2 pts** Incorrect

✓ - **1 pts** Blank

4.7 1 / 2

- **0 pts** Correct
- **2 pts** Incorrect

✓ - **1 pts** Blank

4.8 2 / 2

✓ - **0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

4.9 2 / 2

✓ - **0 pts** Correct
- **2 pts** Incorrect
- **1 pts** Blank

Math 61, Winter 2020
Introduction to Discrete Structures
Midterm Exam 2

February 21, 2020

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient work/explanations, unless the problem explicitly states that no work is necessary. You can not use any notes, books, or electronic devices of any kind during the exam. If you have a question about any particular problem, please raise your hand. At the completion of the exam, please hand in the exam booklet. If you have any questions about the grading of the exam, please consult the midterm regrade policy in the course syllabus.

Name: Justin Chao

UID: 705312352

I certify that the work appearing on this exam is completely my own.

Signature: ____________________

Question:	1	2	3	4	Total
Points:	15	10	15	18	58
Score:					

1. Do the following:

   (a) Suppose $f \subseteq X \times Y$ is a relation. State in precise mathematical language (for instance, using quantifiers) what it means for f to be a *function* $f : X \to Y$. (5)

   (b) Suppose $f : X \to Y$ is a function. State separately (again, in precise mathematical language) what it means for f to be (i) an *injection* and (ii) a *surjection*. (5)

   (c) Suppose $f : X \to Y$ is a function. Recall that an *inverse* of f is a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Suppose $g, h : Y \to X$ are both inverses of f. Prove that $g = h$. Justify all steps. (5)

a) A function $f: X \to Y$ is a relation where
(i) $\forall x \in X \exists y \in Y ((x,y) \in f)$
(ii) $\forall x \in X \forall y_1, y_2 \in Y ((x,y_1) \in f \wedge (x,y_2) \in f \to y_1 = y_2)$

b) (i) $f: X \to Y$ is an injection if
$\forall x_1, x_2 \in X (f(x_1) = f(x_2) \wedge x_1 \neq x_2)$
(ii) $f: X \to Y$ is a surjection if
$\forall y \in Y \exists x \in X (f(x) = y)$

c) Sps $f: X \to Y$ and $g, h: Y \to X$ are both inveses to f.
We want to prove $g = h$.

$g = g \circ \mathrm{id}_Y$ (by indentity function propety)
$= g \circ (f \circ h)$ (since h is an inverse of f)
$= (g \circ f) \circ h$ (function composition is associative)
$= \mathrm{id}_X \circ h$ (since f is an inverse of g)
$= h$ (by identity function property)

Thus $g = h$.

2. Do the following:
   (a) State *Euclid's Lemma.* (5)
   (b) Let $a, b, c \in \mathbb{Z}$ and $n \geq 1$ be arbitrary. Suppose $ac \equiv bc \pmod{n}$ and $\gcd(c, n) = 1$. Prove that $a \equiv b \pmod{n}$. (5)

(a) Euclid's Lemma:

Suppose $\exists\, a, b, c \in \mathbb{Z}$ such that $(a,b) \neq (0,0)$ and $\gcd(a,b) = 1$. If $a \mid bc$, then $a \mid c$.

(b) $\gcd(c,n) = 1$

$ac = nq_0 + r$ $\quad bc = nq_1 + r$ $\quad 0 \leq r < n$ $\quad cx + ny = 1$

$nq_0 = ac - r$ $\quad nq_1 = bc - r$

$n \mid ac - r$ $\quad n \mid bc - r$

PROOF: Sps $\gcd(c,n) = 1$ and $ac \equiv bc \pmod{n}$. By definition of (mod n) and quotient remainder theorem $\exists\, q_0, q_1 \in \mathbb{Z}$ s.t.

$ac = nq_0 + r \wedge bc = nq_1 + r$ (with r the same by $ac \equiv bc \pmod{n}$)

Rearrange these we get

$nq_0 = ac - r \wedge nq_1 = bc - r$

so

$nq = ac - r$

$nq + r = ac$

$n \mid (ac - r)$ $\quad n \mid (bc - r)$.

$* n' = n + r$

If we take out the remainder, $n' \mid ac$ and $n' \mid bc$. By euclid's lemma, $n' \mid a$ and $n' \mid b$. Thus $n \mid a + r$ and $n \mid b + r$, so

$a \equiv b \pmod{n}$.

[redacted]

3. Do the following:

(a) Suppose at a super market there are 2 check-out lines (i.e., queues) and 50 customers. How many ways can these 50 customers get into 2 lines? Here the lines and customers are distinguishable and their order in each line matters. Also, lines are allowed to be empty. For full points, your final answer cannot contain a summation (i.e., no "$\sum$" or "$+\cdots+$".) (5)

(b) Use the Euclidean Algorithm to compute $\gcd(77, 53)$. (5)

(c) Use part (b) to find integers $x, y \in \mathbb{Z}$ such that $77x + 53y = \gcd(77, 53)$. (5)

a) $\boxed{\sum_{n=0}^{50} n!(50-n)!}$

$50! + 49! + 2!48! + 3!47!$

$2(50! + 49! +$

b) $\gcd(77, 53)$:

$77 = 53(1) + 24$

$53 = 24(2) + 5$

$24 = 5(4) + 4$

$5 = 4(1) + 1 \leftarrow$

$4 = 1(4) + 0$

$\boxed{\gcd(77, 53) = 1}$

$77 - 53 = 24$ $\quad$ $53 - 48 = 5$

c) $1 = 5 + 4(-1)$

$= 5(1) + (24 + 5(-4))(-1)$

$= 5(5) + 24(-1)$

$= (53 + 24(-2))(5) + 24(-1)$

$= 53(5) + 24(-11)$

$= 53(5) + (77 + 53(-1))(-11)$

$= 53(16) + 77(-11)$

$53 \times 16 = 318 + 530 = 848$ $\quad$ $77 \times 11 = 77 + 770 = 847$

$\boxed{x = -11, \; y = 16}$

4. For each of the following statements, indicate whether they are True or False. **A blank answer will receive 1 point.** [Recall: *True* means the same thing as "always true" and *False* means the same thing as "there exists a counterexample".] No work is necessary for this problem.

(a) __T__ Suppose a group of 100 people want to form a committee of 10 people. There are $\binom{100}{90}$ ways to do this. (2)

$\frac{100}{90!(10!)}$ $\frac{100}{10!(90)}$

(b) __F__ Suppose $a, b, c \in \mathbb{Z}$. If $a|bc$, then either $a|b$ or $a|c$. (2)

$4 | 2 \times 6$

(c) __T__ Suppose $f, g, h, i : X \to X$ are functions such that $f \circ g \circ h \circ i : X \to X$ is surjective. Then $f \circ g : X \to X$ is surjective. (2)

(d) __F__ Suppose $a, b \in \mathbb{Z}$ and $(a, b) \neq (0, 0)$. If $\gcd(a, b) = 1$, then $\gcd(a + b, a - b) = 1$. (2)

$7, 5$ $7+5$ $7-5$ 12 2

(e) ______ Suppose X, Y are finite sets with $|X| = n$ and $|Y| = k$. Then the total number of functions $X \to Y$ is n^k. (2)

(f) ______ Suppose n is an odd positive integer. Then (2)

$$\left|\{(x_1, x_2) \in \mathbb{N}^2 : x_1 + x_2 = n\}\right| \,\%\, 2 \;=\; 0.$$

(g) ______ Suppose $x, y \in \mathbb{Z}$ are arbitrary, p is a prime number and $n \geq 1$ is arbitrary. Then $(x + y)^{p^n} \equiv x^{p^n} + y^{p^n} \pmod{p}$. (2)

(h) __F__ Suppose X, Y, Z are *finite* sets and $f : X \to Y$ and $g : Y \to Z$ are functions. If f is injective and g is surjective, then $|X| \leq |Z|$. (2)

$|X| \leq |Y|$ $|Z| \leq |Y|$

(i) __T__ Suppose X is a finite set. Then $|X| < |\mathcal{P}(X)|$. (2)

$n < 2^n$