Math 61
Fall 2017
10/23/17
Time Limit: 50 Minutes

Name (Print): ____________________
SID Number: ____________________
Section: ____________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a result from class, discussion, or homework you must indicate this** and explain why the result may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	2	
2	3	
3	2	
4	3	
5	2	
6	3	
Total:	15	

1. (2 points) Let A and B be sets. Does the following identity hold?

$$\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$$

If it holds, provide a proof. If it doesn't hold, give a counter-example. (Here, $\mathcal{P}(X)$ denotes the power set of X.)

Solution: This identity doesn't hold in general. For example, if $A = \{1\}$, $B = \{2\}$ then $\{1, 2\}$ is an element of $\mathcal{P}(A \cup B)$ but not of $\mathcal{P}(A) \cup \mathcal{P}(B)$.

2. (3 points) Prove that

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$$

for all integers $n \geq 2$.

Solution: Let $S(n)$ be the statement that the inequality in the problem holds. We will prove that $S(n)$ for $n \geq 2$ by induction.

- For the basis step let $n = 2$. The left hand side is $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$ which is smaller than the right hand side which is $\frac{4}{3}$.
- For the inductive step, assume $S(n)$ for some $n \geq 2$. This implies the first inequality in the following string of inequalitites:

$$\begin{aligned}\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} + \frac{n+1}{n+2} &< \frac{n^2}{n+1} + \frac{n+1}{n+2} \\ &= \frac{n^2(n+2) + (n+1)^2}{(n+1)(n+2)} \\ &< \frac{n(n+1)^2 + (n+1)^2}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{n+2}\end{aligned}$$

  The second inequality follows from $n(n+2) = n^2 + 2n < n^2 + 2n + 1 = (n+1)^2$. Thus $S(n+1)$ holds assuming $S(n)$.

By the Principle of Mathematical Induction, $S(n)$ holds for all $n \geq 2$.

3. (2 points) Let s be the sequence

$$s_n = \prod_{i=1}^{n} i^{(-1)^i}$$

for all positive integers n. What are s_1, s_2, s_3, s_4? Is s increasing? Decreasing? Nonincreasing? Nondecreasing?

Solution:

1. $s_1 = 1$, $s_2 = 2$, $s_3 = 2/3$, $s_4 = 8/3$.
2. s is neither increasing nor decreasing nor nonincreasing nor nondecreasing.

4. (3 points) For this problem recall that a function is a particular type of relation. Let X be a set of n elements ($n \geq 1$). Describe all functions $f : X \to X$ which are also equivalence relations. For each such function, determine the number of equivalence classes.

Solution:

1. Any equivalence relation is reflexive which implies that for all $x \in X$, $(x, x) \in f$. It follows that f is the identity function, i.e. $f(x) = x$ for all $x \in X$. Conversely, $\{(x, x) \mid x \in X\}$ is an equivalence relation on X as it is symmetric and transitive.

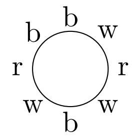
   Concluding, there is a unique function which is also an equivalence relation, namely the identity function.

2. The identity function on a set of n elements has n equivalence classes.

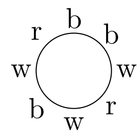
5. (2 points) Two dice are rolled, one blue and one red. How many outcomes have either the blue die 3 or an even sum or both?

Solution: There are 6 outcomes with the blue die 3, there are 18 outcomes with even sum (namely half of 36), and there are 3 outcomes with blue die 3 *and* an even sum. By the inclusion-exclusion principle, the number of outcomes with blue die 3 or even sum is $6 + 18 - 3 = 21$.

6. (3 points) There are 2 red balls, 3 white balls, and 3 blue balls. It is understood that balls of the same color are indistinguishable. The balls are to be arranged in a circle, for example like this:



(a) In how many ways can they be arranged if we don't distinguish between arrangements obtained from each other by rotation? I.e. we don't distinguish for example between the arrangement above and the following one:



(b) What if we add the condition that the red balls are to be placed next to each other?

Solution:

1. If we distinguished arrangements obtained from each other by rotation, there would be $\frac{8!}{2!3!3!}$ many possibilities. To get the correct number we therefore divide by 8:

$$\frac{8!}{2!3!3!8} = 70$$

2. We can think of the two red balls as one single red ball so the answer is similarly to before:

$$\frac{7!}{3!3!7} = 20$$