1. (20 points) Let A, B and C be three sets that are contained within the universal set $\mathcal{U}$, created by Cantor the Mad Setter.
   (a) (10 points) Cantor asks the young mathematician, Alice, if the union $A \cup B \cup C$ may be expressed as
   $$\left(A \cap (B \cup C^c)^c\right)^c.$$
   How should Alice respond? (i) She should say *YES*, or, (ii) she should say *NO*. In each case, you must justify her answer.
   (HINT: If your answer is *YES*, then you are claiming the two sets are equal. If you want to prove two sets, S_1 and S_2 are equal, then you must prove "$S_1 \subseteq S_2$ and $S_2 \subseteq S_1$". If your answer is *NO*, you may point out to a subset of one of these sets that is not contained in the other.)

If $A \cup B \cup C = (A \cap (B \cup C^c)^c)^c$, then ① $A \cup B \cup C \subseteq (A \cap (B \cup C^c)^c)^c$
and ② $(A \cap (B \cup C^c)^c)^c \subseteq A \cup B \cup C$

Let $\mathcal{U} = \mathbb{Z}^+$, $A = \{1,2,3\}$, $B = \{3,4,5\}$, $C = \{2,3,6\}$

$C^c = \{1,4,5,7,\ldots\}$

$B \cup C^c = \{1,3,4,5,7,\ldots\}$ (basically $\mathbb{Z}^+$ without 2, 6)

$(B \cup C^c)^c = \{2,6\}$ (elements exclusive to set C)

$A \cap (B \cup C^c)^c = \{2\}$

$(A \cap (B \cup C^c)^c)^c = \{1,3,4,\ldots\}$ ($\mathbb{Z}^+$ without 2)

$S_1 = A \cup B \cup C = \{1,2,3,4,5,6\}$ $S_2 = (A \cap (B \cup C^c)^c)^c = \{1,3,4,5,\ldots\}$

No, the sets are <u>not equal</u>. The element 2 is included in S_1, but is not included in S_2. Additionally, the subset $(A \cup B \cup C)^c = \{7,8,9\ldots\}$ is included in S_2, but not included in S_1.

(b) (5 points) Let $E = \{\text{Feanor, Finarfin, Fingolfin}\}$. Which one is true? Briefly explain.

(i) Fingolfin $\in \mathcal{P}(E)$
(ii) $\{\text{Fingolfin}\} \in \mathcal{P}(E)$
(iii) $\{\text{Fingolfin}\} \subseteq \mathcal{P}(E)$

A power set P(E) includes all possible subsets of E.

(i) False (ii) True (iii) False

(i) is false because Fingolfin is an element of set E, but only subsets are elements of P(E).

(ii) is true because {Fingolfin} is a subset of E, and by definition a power set includes all subsets of a set. Thus, {Fingolfin} is an element of P(E).

(iii) is false because the element Fingolfin is not a subset and therefore not included in P(E). Thus, {Fingolfin} is not a subset of P(E).

(c) (5 points) How many elements does $\mathcal{P}(\mathcal{P}(E))$ have?

★The power set of a set with n elements has 2^n elements.

$P(E) = 2^3 = 8$ elements

$P(P(E)) = 2^8 = \boxed{256 \text{ elements}}$

2. (10 points) Smeagol and Gollum are two fishermen. Smeagol catches 4^n and Gollum catches n^2 fish in n hours, where n may be any positive integer.

   Prove that $\forall n \in \mathbb{Z}^+$ Smeagol catches more fish than Gollum in n hours.

Prove that $\forall n \in \mathbb{Z}^+$, $\underbrace{4^n}_{\text{Smeagol}} > \underbrace{n^2}_{\text{Gollum}}$.

① Let $n = 1$ (base case):

$4^1 > 1^2 \rightarrow 4 > 1$ ✓

② Let $n = k$ where $k \in \mathbb{Z}^+$. Assume true for $n = k$.

$4^k > k^2$

③ Let $n = k+1$:

Claim: $4^{k+1} > (k+1)^2 = k^2 + 2k + 1$

$4^{k+1} = 4 \cdot 4^k$

$4^{k+1} = 2 \cdot 4^k + 4^k + 4^k$

$4^{k+1} = 4^k + 4^k + 4^k + 4^k$

$4^{k+1} > k^2 + k^2 + k^2 + k^2$ (by assumption from ②)

$4^{k+1} > k^2 + k + k + k$ (by assumption $k^2 > k$ for all $k \in \mathbb{Z}^+$)

$4^{k+1} > k^2 + k + k + 1$ (by assumption $k \in \mathbb{Z}^+$)

$4^{k+1} > k^2 + 2k + 1$

$4^{k+1} > (k+1)^2$

$\therefore 4^{k+1} > (k+1)^2$ for all $k \in \mathbb{Z}^+$ is true using proof by induction.

3. (30 points)

(a) (10 points) Let $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ be the relation defined by

$$(x, y) \in R \iff 3 | (x + 5y).$$

→ $(x+5y)$ is divisible by 3

Is R an equivalence relation? Justify your answer.

Equivalence = reflexive ①, symmetric ②, transitive ③

① Reflexive : $(x,x) \in R$ for every $x \in \mathbb{Z}^+$

Let (x_1, x_1) be any value such that $x_1 \in \mathbb{Z}^+$.

$(x_1 + 5x_1) = 6x_1$ ← always divisible by 3 to produce $2x_1$

∴ R is reflexive.

② symmetric : If $(x,y) \in R$, then $(y,x) \in R$ for all $x, y \in R$

Let $(x,y) = (5, 26) \rightarrow 5 + 5(26) = 135$ (divisible by 3)

$(y,x) = (26, 5) \rightarrow 26 + 5(5) = 51$ (not divisible by 3)

∴ R is not symmetric as there exists (x,y) pairs in R that don't have (y,x) included in R.

③ Transitive : If $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$

Let $(x,y) \in R$ such that $(x+5y)$ is divisible by 3. Let $(y,z) \in R$ such that $(y+5z)$ is divisible by 3.

$x + 5y = 3K$ K = constant

$y + 5z = 3L$ L = constant

$x + 6y + 5z = 3(K+L)$ addition of two equation

$x + 5z = 3(K+L-2y)$ $(K+L-2y)$ is some integer value

∴ Given that $(x,y) \in R$ and $(y,z) \in R$, (x,z) is divisible by 3 and thus $(x,z) \in R$.

R is transitive.

R is not a equivalence relation. R is reflexive and transitive but not symmetric.

(b) (20 points) Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. Let

$$R = \{(1, a), (1, b), (2, b), (3, a)\} \subseteq X \times Y$$

and

$$S = \{(a, 1), (a, 3), (b, 2)\} \subseteq Y \times X.$$

Is $S \circ R$ reflexive? Symmetric? Anti-symmetric? Transitive?

(HINT: You may use the matrix representation of $S \circ R$.)

$S \circ R = \{(1,1), (1,3), (1,2), (2,2), (3,1), (3,3)\}$

① $S \circ R$ is reflexive as it includes (x, x) for all values in set X.
• $(1,1), (2,2), (3,3)$

② $S \circ R$ is not symmetric as it includes $(1,2)$ but not $(2,1)$.

③ $S \circ R$ is not anti-symmetric as it includes both $(1,3)$ and $(3,1)$, but $1 \neq 3$.

④ $S \circ R$ is not transitive as it includes $(3,1)$ and $(1,2)$, but not $(3,2)$.

4. (20 points) Let $\mathbb{R}$ be the set of all real numbers.

(a) (10 points) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the function defined by

$$f(x, y) = (x + y, x - y).$$

Is f a bijection? If your answer is *YES*, you must prove your claim. If your answer is *NO*, you should prove that at least one of the conditions for a function being a bijection is violated.

Bijection: both one-to-one ① and onto ②

① One-to-one: if $f(x_1, y_1) = f(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$

suppose $f(x_1, y_1) = f(x_2, y_2)$

$$(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$$

$$x_1 + y_1 = x_2 + y_2 \quad ①$$

$$x_1 - y_1 = x_2 - y_2 \quad ②$$

$$2y_1 = 2y_2 \to y_1 = y_2 \quad \text{(subtract ② from ①)}$$

$$2x_1 = 2x_2 \to x_1 = x_2 \quad \text{(add ① and ②)}$$

$\therefore$ if $f(x_1, y_1) = f(x_2, y_2)$, then $x_1 = x_2$ and $y_1 = y_2$

f is <u>injective</u> (one-to-one)

② onto: for every $f(x, y) \in \mathbb{R} \times \mathbb{R}$, there exists $x_1 \in \mathbb{R}$ and $y_1 \in \mathbb{R}$ such that the function equals $f(x, y)$.

Let (a, b) be the result of $f(x+y, x-y)$ for $(x, y) \in \mathbb{R} \times \mathbb{R}$

$$\begin{array}{l} a = x + y \\ b = x - y \end{array} \to \begin{array}{l} 2x = a + b \to x = \frac{1}{2}(a+b) \\ 2y = a - b \to y = \frac{1}{2}(a-b) \end{array}$$

Given that $(a, b) \in \mathbb{R} \times \mathbb{R}$, the sum of any 2 real numbers divided by 2 will yield a real number $x \in \mathbb{R}$. The difference of two real numbers divided by two will also yield a real number $y \in \mathbb{R}$.

$\therefore$ f is <u>surjective</u>, as for every $f(x, y) \in \mathbb{R} \times \mathbb{R}$, there exists an $(x_1, y_1) \in \mathbb{R}$ such that $f(x, y) = (x_1 + y_1, x_1 - y_1)$.

<u>f is a bijection</u> as it is both injective and surjective.

function: each x maps to exactly 1 y

(b) (10 points) Recall the relations R, S and $S \circ R$ in question 3(b). Are these relations also functions? Explain briefly.

R is not a function as the x-value of 1 maps to two different y values : (1, a) and (1, b).

S is not a function as the x-value of a maps to two different y values : (a, 1) and (a, 3).

S∘R is not a function as the x-value of 1 maps to three different y values : (1,1), (1,2), and (1,3).

5. (20 points)

(a) (10 points) Elros is a half-elven lord who has four children: Vardamir, Manwendil, Atanalcar and Tindomiel. Elros would like to distribute his 20 sapphire gemstones to his children so that all children gets at least one gemstone. Vardamir, being the eldest, must get at least 3, and Tindomiel, being the only daughter, should get at least 4. In how many ways can Elros achieve this?

20 total gemstones
− 4 goes to Tindomiel
− 3 goes to Vardamir
− 2 goes to remaining 2 children

11 ← total gemstones left to distribute

4 children = 3 dividers (to split into 4 parts)
11 gemstones + 3 dividers = 14 total blanks to fill

Number of ways $= \binom{14}{3} = \frac{14!}{(11!)(3!)} = {}_{14}C_3 = \boxed{364 \text{ ways}}$

(b) (10 points) Turin enjoys gambling. He tosses a coin 10 times, where each outcome is H or T. How many outcomes are there so that there are at most 8 H?

(HINT: The order matters, $HHHHHHHHHT$ is a different outcome than $THHHHHHHHH$.)

Total # outcomes = 2^{10} = 1024

At most 8H = 1024 − # outcomes 9H − # outcomes 10H

9H : ${}_{10}C_{9}$ = 10 outcomes

10H : 1 outcome (all heads)

At most 8H : 1024 − 10 − 1 = $\boxed{\text{1013 outcomes}}$