# Final Exam

MATH 61 @ UCLA (Summer 2021)

Assigned: July 28, 2021.

## Instructions / Admonishment

1. SHOW ALL WORK.
   A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit.
2. Duration: 24 hours.
3. The following is my own work, without the aid of any other person.
   Signature: ____________________________________________

**Problem 1 Truth tables.**

Write the truth table of the following propositions:

(i) $(p \wedge q) \vee (\overline{p} \vee q)$

(ii) $(\overline{p \wedge q}) \vee (r \wedge \overline{p})$

**Problem 2 Proofs.**

Let A and B be two sets. Let $\overline{\mathrm{B}}$ be the complement of B. Prove that the following statements are equivalent.

(i) $\mathrm{A} \subseteq \mathrm{B}$

(ii) $\mathrm{A} \cap \overline{\mathrm{B}} = \emptyset$

(iii) $\mathrm{A} \cup \mathrm{B} = \mathrm{B}$

**Problem 3 Relations.**

Let $\mathrm{A} = \{1, 2, 3\}$, $\mathrm{B} = \{\mathrm{x}, \mathrm{y}\}$, and $\mathrm{C} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$. Let define the relation $\mathscr{R}_1$ from A to B by $\mathscr{R}_1 = \{(1, x), (2, x), (2, y), (3, y)\}$. Let define the relation $\mathscr{R}_2$ from B to C by $\mathscr{R}_2 = \{(x, a), (x, b), (y, a), (y, c)\}$.

(i) Find the matrix $\mathrm{A}_1$ of the relation $\mathscr{R}_1$ relative to the orderings $1, 2, 3$ and $x, y$.

(ii) Find the matrix $\mathrm{A}_2$ of the relation $\mathscr{R}_2$ relative to the orderings $x, y$ and $a, b, c$.

(iii) Find the matrix product $\mathrm{A}_1\mathrm{A}_2$. What is the matrix of the relation $\mathscr{R}_2 \circ \mathscr{R}_1$?

**Problem 4 Multi- and bi-nomial Coefficients.**

(i) Find the coefficients of $x^3yz^4$ in the expression of $(2x+y+z)^8$

(ii) Use the Binomial Theorem to prove that $\sum_{k=0}^{n} 2^{n-k}(-1)^k\binom{n}{k} = 1$

**Problem 5 A bit strings sequence.**

Let $C_n$ be the number of bit strings of length $n$ that contain three consecutive 0's.

(i) Write a recurrence relation that defines the sequence $C_n$. (Hint: count the number of this type of strings ending with 1, or with 10, or with 100, or with 000)

(ii) What are the initial conditions?

(iii) How many bit strings of length seven contain three consecutive 0's?.

**Problem 6 The Lucas sequence.**

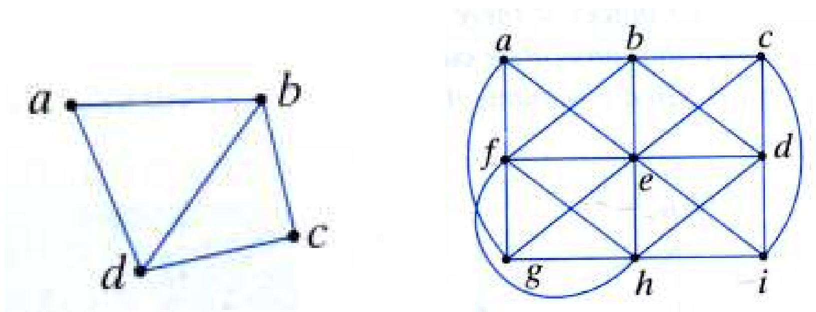
The Lucas sequence ( named after Edouard Lucas, the inventor of the Tower of Hanoi puzzle) is defined by $L_n = L_{n-1} + L_{n-2}$, $n \geq 3$; $L_1 = 1$, $L_2 = 3$.

(i) Find the values $L_3, L_4, L_5$.

(ii) Show that $L_{n+2} = f_{n+1} + f_{n+3}$, $n \geq 1$ where $f_1, f_2, f_3, \cdots$ is the Fibonacci sequence.

**Problem 7 Graph theory.**

(i) Explain why none of the graphs in Figure 1(a) and 1(b) has a path from the node (vertex) $a$ to $a$ that passes through each edge exactly one time.




(a) (b)

Figure 1: Some sample of graphs

(ii) Decide whether the graph in Figure 2 has an Euler cycle. If the graph has an Euler cycle, exhibit one.

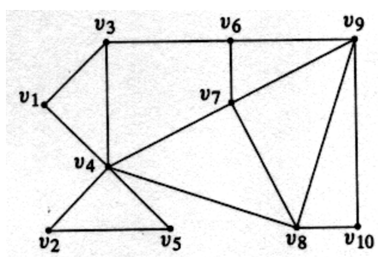



Figure 2: Graph with an Euler cycle or not