1. (5 points) Let p, q and r be three propositions. Define

$$P = (p \implies q) \implies r$$

and

$$Q = p \implies (q \implies r).$$

Are P and Q logically equivalent? Justify your answer, using a truth table.

p	q	r	$(p \to q)$	$(q \to r)$	$(p \to q) \to r$	$p \to (q \to r)$
F	F	F	T	T	F	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

P and Q are not logically equivalent. They differ in the above two circled cases: ① $p = F, q = F, r = F$ and ② $p = F, q = T, r = F$

2. (10 points) Consider the set $S = \{1, 2, 3, \ldots, 10\}$. Let R and P be two relations defined on S.

   If R and P are both antisymmetric, does that imply $R \cap P$ is an antisymmetric relation? Prove or give a counter example.

Antisymmetric : if $(x,y) \in R$ and $(y,x) \in R$, then $x = y$.
if $x \neq y$, then $(x,y) \notin R$ or $(y,x) \notin R$

proof: Given that R & P are both antisymmetric, the two relations may include $(x,y) \in S \times S$ OR $(y,x) \in S \times S$ or neither, but not both.

$R \cap P$ includes all (x,y) such that $(x,y) \in R$ and $(x,y) \in P$.
If $(x,y) \in R \cap P$, then $(y,x) \notin R \cap P$ as both sets cannot contain (x,y) and (y,x) if $x \neq y$ (definition of antisymmetric).

$(x,y) \in R \cap P \rightarrow (y,x) \notin R \cap P$

$\therefore$ $R \cap P$ is antisymmetric given R & P are antisymmetric

3. (10 points) Let X, Y, Z be any nonempty sets, $g : X \to Y$ and $f : Y \to Z$ be two functions. Prove or give a counter example to the following statement

$f \circ g = f(g) = X \to Z$

"If g is onto, then $f \circ g$ is onto."

Onto : For every $y \in Y$, there exists $x \in X$ such that $f(x) = y$

Let $X = \{0,1,2\}$, $Y = \{1,2,3\}$, $Z = \{2,3,4\}$

$g = \{(0,1),(1,2),(2,3)\}$ (g is surjective)

$f = \{(1,2),(2,2),(3,4)\}$ (f is not surjective)

$f \circ g = \{(0,2),(1,2),(2,4)\}$

Given that g is onto, $f \circ g$ is not onto as there is no x-y pair mapping an x-value to the z-value of 3.

$\therefore$ The statement is <u>false</u>.

4. (5 points) Let $X = \{1, 2, 3\}$. How many bijections $f : X \to X$ are there from such that

$$\forall x \in X, \quad f(x) \neq x$$

is satisfied? Justify your answer.

All bijections:

$\{(1,1), (2,2), (3,3)\}$

$\{(1,1), (2,3), (3,2)\}$

$\{(1,2), (2,1), (3,3)\}$

$\{(1,2), (2,3), (3,1)\}$

$\{(1,3), (2,1), (3,2)\}$

$\to$ only two bijections satisfy $\forall x \in X,\ f(x) \neq x$

$\boxed{2 \text{ bijections}}$

5. (10 points) An inventory consists of an ordered list of k items that are marked as "available" and 5 items that are marked as "unavailable". What is the smallest value of $k \in \mathbb{Z}^+$ such that we are certain that at least two items that are marked as available will be exactly three items apart in this list? Justify your answer. (HINT: Three apart means $A_i * * A_j$, where each $*$ could be either A_l or U_l)

Using the pigeonhole principle :

_ O _ O _ O _ O _ O _

holes

• Given 5 "unavailable" items, there are 6 "holes" to fill

• 6 = highest possible number of available items where there exists a possibility of two "A" not being 3 items apart

inserting A anywhere in these 4 lists will be valid

K = 6 : A U A A U A U A A U A A U A OR A A A U U U A V A U A OR

A U A V A U U U A A A OR A A A U U U U U A A A

By the first form of the pigeonhole principle, if the number of available items exceeds the number of holes, some pigeonhole will contain at least two available items.

K = 7 : A A U A U A U A U A U A

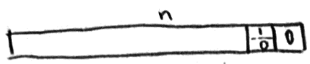
In this case, if any hole exceeds 1 available item, then there are two As that are 3 items apart no matter which hole recieves 2 A items.

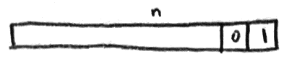
$\boxed{\therefore\ K = 7}$

6. (10 points) Let s_n be the number of n-bit strings of 0's and 1's that avoid the pattern "11". Find the recurrence relation for s_n and find the solution for this recurrence relation.

For any given n, let a_n = total strings w/out "11", ending in "0" and
b_n = total strings w/out "11", ending in "1"

such that $S_n = a_n + b_n$





since the n-1 value can either be 0 or 1,

$a_n = a_{n-1} + b_{n-1}$

valid strings for n-1 ending in 1 or 0

the n-1 value must be 0 for the string to be valid:

$b_n = a_{n-1}$

$$S_n = a_n + b_n = (a_{n-1} + b_{n-1}) + a_{n-1}$$
$$= S_{n-1} + a_{n-1}$$

(a_n can be rewritten as S_{n-1})

$$\boxed{S_n = S_{n-1} + S_{n-2}}$$ ← Recurrence relation

n	S_n	possible strings
0	0	
1	2	"0", "1"
2	3	"00", "01", "10"
3	5	
4	8	
5	13	
⋮	⋮	

0	0
1	1
2	1
3	2
4	3

This relation is essentially the fibonacci sequence but n is shifted down by 2.

Solution:

$$S_n = (1)(S_{n-1}) + (1)(S_{n-2})$$
$$a_n = (1)a_{n-1} + (1)a_{n-2}$$

$c_1 = 1$, $c_2 = 1$

$$t^2 - t - 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{1 - 4(1)(-1)}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}$$ (Theorem 7.2.11)

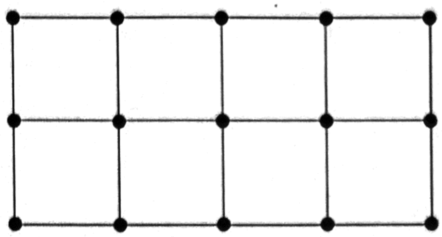
$$S_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$$ ← Given $S_0 = 0$, $b = -d$

$$S_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n - b\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\boxed{S_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\sqrt{5}}} \quad \boxed{\text{for } n > 0}$$

SOLUTION ↑

7. (10 points) Consider the rectangular grid graph, $R_{a,b}$, with a rows and b columns of vertices (with ab vertices in total). For example $R_{3,5}$ is as follows:



(a) (5 points) Identify *all* pairs $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ (the whole subset satisfying this) s.t. $R_{a,b}$ has a Hamiltonian path. Recall that a simple path that goes over every vertex exactly once is a Hamiltonian path.

$\boxed{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+}$

All $R_{a,b}$ within the above constraint has a Hamiltonian path.

(b) (5 points) Identify *all* pairs $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ (the whole subset satisfying this) s.t. $R_{a,b}$ has an Euler path or an Euler cycle. Recall that in a connected graph, if a path goes over every edge exactly once then it is called an Euler path, and if a cycle goes over every edge exactly once, then it is called an Euler cycle.

★ Grid contains Euler path/cycle if all vertices have even degree, or contains at most 2 vertices with odd degree.

For any grid graph $R_{a,b}$, the number of vertices with an odd degree is $V_{odd} = 2(a-2) + 2(b-2)$ as vertices on the edges (excluding the corners) will have degrees of 3. The exceptions are where $a = 1$ or $b = 1$, as grids with only 1 row/column will always contain a Euler path.

$2(a-2) + 2(b-2) \leq 2$ (at most 2 vertices of odd degree)

$2a + 2b - 8 \leq 2$

$2a + 2b \leq 10$

$a + b \leq 5$

contains Euler path/cycle:
$R_{a,b}$ given that $a+b \leq 5$ OR
$R_{a,b}$ where $a = 1$ or $b = 1$

8. (10 points) Let A be the adjacency matrix of a simple, weighted graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$. Define the *weighted adjacency matrix* of G, denoted W, to be the $n \times n$ matrix such that $\forall i, j \in \{1, 2, \ldots, n\}$,

$$W_{ij} = 0 \iff A_{ij} = 0,$$

and whenever $W_{ij} \neq 0$, it is set to be the nonnegative weight associated with the edge between v_i and v_j. Shortly put, W is an adjacency matrix in which all edges are represented with their weights, instead of 1's.

Let G be the simple, weighted graph with the following weighted adjacency matrix W

$$W = \begin{array}{c} \\ S \\ A \\ B \\ C \\ D \\ F \\ G \\ H \\ E \end{array} \begin{array}{c} \begin{array}{ccccccccc} S & A & B & C & D & F & G & H & E \end{array} \\ \begin{bmatrix} 0 & 3 & 7 & 5 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 7 & 0 & 0 & 0 & 0 \\ 7 & 1 & 0 & 3 & 2 & 1 & 3 & 0 & 0 \\ 5 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 7 & 2 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 3 & 3 & 2 \\ 0 & 0 & 3 & 2 & 0 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 & 5 & 0 \end{bmatrix} \end{array}.$$

Note that the rows and columns of W are ordered according to the following order of vertices:

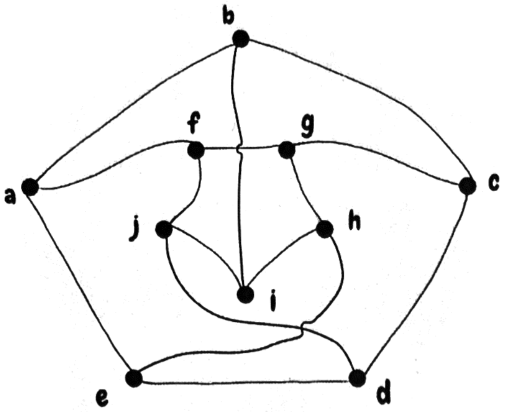
$$S, A, B, C, D, F, G, H, E.$$

Using Dijkstra's shortest path algorithm, you are to find the shortest path from S to E. Recall that *shortest path* meant the path with the minimum total sum of weights. Find the answer by filling in the following table correctly, presenting how each iteration will proceed.

Iteration	Processed Vertex	$L(S)$	$L(A)$	$L(B)$	$L(C)$	$L(D)$	$L(F)$	$L(G)$	$L(H)$	$L(E)$
0	N/A - initialization	0	∞	∞	∞	∞	∞	∞	∞	∞
1	S	0	3	7	5	∞	∞	∞	∞	∞
2	A	0	3	4	5	10	∞	∞	∞	∞
3	B	0	3	4	5	6	5	7	∞	∞
4	C	0	3	4	5	6	5	7	∞	∞
5	F	0	3	4	5	6	5	7	8	7
6	D	0	3	4	5	6	5	7	7	7
7	G	0	3	4	5	6	5	7	7	7
8	H	0	3	4	5	6	5	7	7	7
9	E	0	3	4	5	6	5	7	7	7

$\boxed{\text{Shortest path} = 7}$

9. (10 points) Consider the following graph, G:

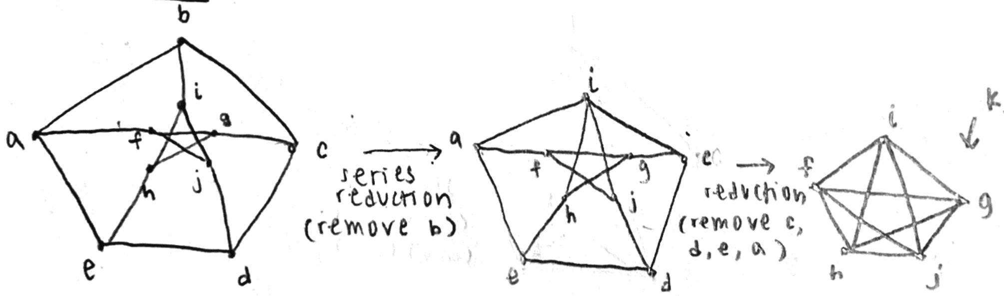


$15 - 10 + 2 = 7$

Is G planar? If so, provide a planar drawing of it. If not planar, prove that it is so, using Kuratowski's Theorem.

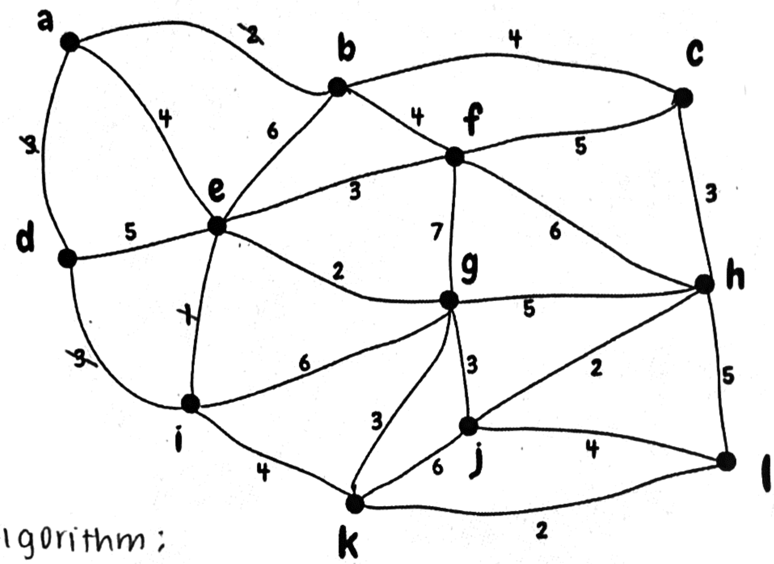
G is not planar:

Redrawn G



Graph G contains the subgraph homeomorphic to K_5, so G is <u>not planar</u> according to Kuratowski's theorem.

10. (10 points) Find a minimal spanning tree for the following weighted tree. Also report the corresponding minimum total weight. (HINT: You may use whichever approach you prefer, but your final answer must be a spanning tree and its total weight.)



Using Prim's Algorithm:

⓪ start at vertex a and add

① Add edge (a, b) with weight 2 & vertex b

② (a, d) with weight 3 & vertex d

③ (d, i) w/ weight 3 & vertex i

④ (i, e) w/ weight 1 & vertex e

⑤ (e, g) w/ weight 2 & vertex g

⑥ (g, k) w/ weight 3 & vertex k

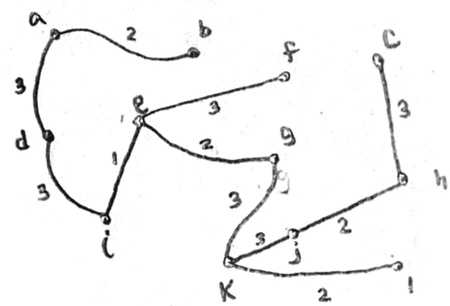
⑦ (k, l) w/ weight 2 & vertex l

⑧ (g, j) w/ weight 3 & vertex j

⑨ (j, h) w/ weight 2 & vertex h

⑩ (e, f) w/ weight 3 & vertex f

⑪ (h, c) w/ weight 3 & vertex c



Minimal spanning Tree

Total weight = $\boxed{27}$

11. (10 points) How many nonisomorphic (free) trees are there on six vertices? Draw all of them.

There are six nonisomorphic trees on six vertices:

1

2

3

4

5

6