Midterm 1 (Version A)

UCLA: Math 61, Winter 2018

Instructor: Jens Eberhardt
Date: 02 February 2017

- This exam has 4 questions, for a total of 36 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.

Name: [redacted]

ID number: [redacted]

Discussion section (please circle):

Day/TA	HUNT, CHRISTOPHER	HAN, KYUTAE	MENEZES, DEAN
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	12	10
2	10	10
3	6	6
4	8	8
Total:	36	34

Please note! The following two pages will not be graded. You must indicate your answers **here** *for them to be graded!*

Question 1.

Part	A	B	C	D
(a)				D
(b)	A			
(c)			C	
(d)			C	
(e)				D
(f)			C	

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) Let $X = \{1,2,3\}$, then $R = \{(1,3),(2,2),(3,1)\}$ is

   A. bijective
   B. reflexive
   C. antisymmetric
   D. not symmetric

   (b) (2 points) Let $X = \mathbb{Z}_{\geq 0} = \{0,1,2,\dots\}$. Define an equivalence relation R on X by:

   $$xRy \text{ if } x - y \text{ is divisible by } 3.$$

   Then the partition $\mathcal{P}_R$ associated to the relation R is:

   A. $\mathcal{P}_R = \{\{0,3,6,\dots\},\{1,4,7,\dots\},\{2,5,8,\dots\}\}$
   B. $\mathcal{P}_R = \{\{0,2,4,\dots\},\{1,3,5,\dots\}\}$
   C. $\mathcal{P}_R = \{\{0\},\{1\},\{2\},\dots\}$
   D. $\mathcal{P}_R = \{(0,0),(0,3),(3,0)\dots,(3,3),(3,6),(6,3),\dots,$
   $(1,1),(1,4),(4,1),\dots,(4,4),(4,7)(7,4),\dots,$
   $(2,2),(2,5),(5,2),\dots,(5,5),(5,8)(8,5)\dots\}$

   (c) (2 points) Define a partition $\mathcal{P}$ on $\{0,1,2,3\}$ by

   $$\mathcal{P} = \{\{0\},\{1,3\},\{2\}\}.$$

   Let $R_{\mathcal{P}}$ be the associated equivalence relation on $\{0,1,2,3\}$. Then

   A. $R_{\mathcal{P}} = \{(0,0),(1,3),(3,1),(2,2)\}$
   B. $R_{\mathcal{P}} = \{\{0\},\{1,3\},\{2\}\}$
   C. $R_{\mathcal{P}} = \{(0,0),(1,1),(2,2),(3,3),(1,3),(3,1)\}$
   D. $R_{\mathcal{P}} = \{0,1,2,3\}$

(d) (2 points) Let $X = \{0,1,2,3\}$. For a set Y denote by $\mathcal{P}(Y) = \{S | S \text{ is a subset of } Y\}$ the power set of X. Then

A. $|\mathcal{P}(X \times X)| = 2 \cdot 16$

B. $|\mathcal{P}(X \times X)| = 2^8$

C. $|\mathcal{P}(X \times X)| = 2^{16}$

D. $|\mathcal{P}(X \times X)| = 2 \cdot 8$

$X \times X$

$\{1,2\}$ $\{1,2\}$

$\{(1,1),(1,2),(2,1),(2,2)\}$

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$|\mathcal{P}(X \times X)| = 2^{16}$

(e) (2 points) Let $n \geq 1$ be a positive integer. Then

$$\prod_{i=1}^{n} 2i$$

is equal to

A. $2^n \frac{n(n+1)}{2}$

B. $\frac{2^n 2^{n+1}}{2}$

C. $(2n)!$

D. $2^n \cdot n!$

$2 \cdot 4 \cdot 6 \cdot 8$

$2, 8, 48, 384$

2^n

(f) (2 points) Let $X = \{a, b\}$. Denote by $X^{\geq 3}$ the set of all strings over X of length bigger or equal than three and by X^* the set of all strings over X. Then

A. $|X^* - X^{\geq 3}| = 5$

B. $|X^* - X^{\geq 3}| = 6$

C. $|X^* - X^{\geq 3}| = 7$

D. $|X^* - X^{\geq 3}| = 8$

$\{\varepsilon, a, b, a^2, b^2, ab, ba\}$

2. In the following questions, simply write down your answer. There is *no justification needed*. Do not simplify expressions as $2^4, 6!, C(n,r), \ldots$.

(a) (2 points) Determine the number of 7-bit strings starting in 1010.

1010 _ _ _

$\boxed{2^3}$

(b) (2 points) Determine the number of 7-bit strings ending in 1010.

_ _ _ 1010

$\boxed{2^3}$

(c) (2 points) Determine the number of 7-bit strings starting in 1010 or ending in 1010.

1010 _ _ _ 2^3

_ _ _ 1010 2^3

$\boxed{2^3 + 2^3}$

(d) (2 points) You have three friends: Rocco, Gina and Hans. And you have seven different sweets: a popsicle, a piece of apple pie, a chocolate bar, a Berliner, a jelly doughnut, a marshmallow and a lemon drop.
You want to give two sweets to Rocco, three to Gina and two to Hans. In how many ways could you do this?

$\boxed{\dfrac{7!}{2!\,3!\,2!}}$

(e) (2 points) Let $X = \{1, 2, 3, 4, 5\}$. Determine the number of elements of the following set

$$\{S \mid S \text{ is a subset of } X \text{ and } |S| = 3\}.$$

=> Subsets of length 3

$\boxed{C(5,3)}$

123 234 345
124 235
125 245
134
135
145

2. (6 points) Prove by induction that

$$(*)\quad \sum_{i=1}^{n}(2i-n)=n$$

for any integer $n \geq 1$.
Hint: Use

$$\sum_{i=1}^{n+1}(2i-(n+1))=\left(\sum_{i=1}^{n+1}(2i-n)\right)-(n+1).$$

1) Basis Step $(n=1)$

$$\sum_{i=1}^{1}(2i-1) = 2(1)-1 = 2-1 = 1 \quad //$$

2) Inductive Step: Assume that the statement $(*)$ is true for all $n \geq 1$. We must show it is also true for $n+1$.

$$\sum_{i=1}^{n+1}(2i-(n+1)) = n + (2(n+1)-(n+1))$$

$$LHS = \left(\sum_{i=1}^{n+1}(2i-n)\right) - (n+1)$$

$$\left(\sum_{i=1}^{n}(2i-n)\right) + 2(n+1)-(n+1) - (n+1)$$

$$= n + (2(n+1)-(n+1)) \quad //$$

By ~~the~~ induction, it's true for all $n \geq 1$

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4. Answer the following questions, *justifying* your answers. (If an answer is Yes, explain why. If an answer is No, give a counterexample.)

(a) (2 points) Define $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$. Is f surjective?

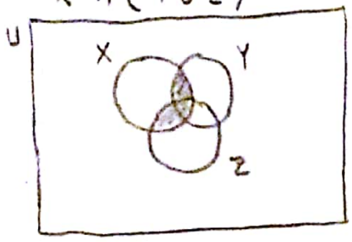
$\boxed{\text{Yes}}$ Since for all $y \in \mathbb{R}_{\geq 0}$, there is an $x = \sqrt{y} \in \mathbb{R}$, $f(x) = (\sqrt{y})^2 = y$. Hence, f is surjective.

(b) (2 points) Let X, Y, Z be sets. Then

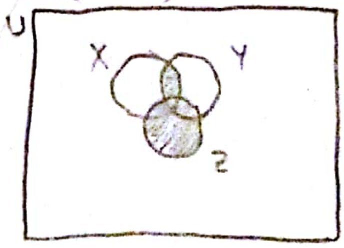
$$X \cap (Y \cup Z) = (X \cap Y) \cup Z.$$

$\boxed{\text{No}}$

$X \cap (Y \cup Z)$



$(X \cap Y) \cup Z$



(c) (2 points) Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Is there a bijective function $f : X \to Y$?

$\boxed{\text{Yes}}$ Let $f : X \to Y$, such that $f = \{(1,a), (2,b), (3,c)\}$. Then, f is injective and surjective. Hence, f is bijective.

(d) (2 points) Let $X = \{a, b, c, \ldots, z\}$ be the alphabet. Let α and β be strings over X. Is

$$\alpha\beta = \beta\alpha?$$

$\boxed{\text{No}}$ Let $\alpha = a$ and $\beta = b$

$ab \neq ba$