Please note! The following two pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				X
(b)				X
(c)		X		
(d)			X	
(e)			X	
(f)				X

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) Let $X = \{0, 1, 2, 3\}$. For a set Y denote by $\mathcal{P}(Y) = \{S | S \text{ is a subset of } Y\}$ the power set of X. Then

   A. $|\mathcal{P}(X \times X)| = 2 \cdot 16$
   B. $|\mathcal{P}(X \times X)| = 2^{16}$
   C. $|\mathcal{P}(X \times X)| = 2 \cdot 8$
   D. $|\mathcal{P}(X \times X)| = 2^8$

   (b) (2 points) Let $n \geq 1$ be a positive integer. Then

   $n = 3$

   $$\frac{3^n 3^{n+1}}{2} \qquad \prod_{i=1}^{n} 3i$$

   $3(3) \cdot 3(2) \cdot 3(1)$

   is equal to

   ~~A.~~ $(3n)!$ $\qquad \frac{3^3 3^4}{2} = \frac{9 \cdot 27}{2}$ $\qquad 9 \cdot 6 \cdot 3 = 162$
   B. $3^n \frac{n(n+1)}{2}$
   C. $\frac{3^n 3^{n+1}}{2}$ $\qquad (3(3))! = 9! \neq 162$
   D. $3^n \cdot n!$

   $3^3 \cdot 3! = 27 \cdot 6 = 162$

   $3^3 \frac{3(3+1)}{2} = 9\left(\frac{12}{2}\right) = 9 \cdot 6 \neq 162$

   (c) (2 points) Define a partition $\mathcal{P}$ on $\{0, 1, 2, 3\}$ by

   $$\mathcal{P} = \{\{0\}, \{1, 3\}, \{2\}\}.$$

   Let $R_{\mathcal{P}}$ be the associated equivalence relation on $\{0, 1, 2, 3\}$. Then

   A. $R_{\mathcal{P}} = \{(0, 0), (1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$
   B. $R_{\mathcal{P}} = \{(0, 0), (1, 3), (3, 1), (2, 2)\}$
   C. $R_{\mathcal{P}} = \{\{0\}, \{1, 3\}, \{2\}\}$
   D. $R_{\mathcal{P}} = \{0, 1, 2, 3\}$

(d) (2 points) Let $X = \{1, 2, 3\}$, then $R = \{(1,3), (2,2), (3,1)\}$ is

A. not symmetric
B. reflexive ?
C. bijective
D. antisymmetric

(e) (2 points) Let $X = \{a, b\}$. Denote by $X^{\geq 3}$ the set of all strings over X of length bigger or equal than three and by X^* the set of all strings over X. Then

A. $|X^* - X^{\geq 3}| = 5$
B. $|X^* - X^{\geq 3}| = 6$
C. $|X^* - X^{\geq 3}| = 7$
D. $|X^* - X^{\geq 3}| = 8$

ab
ba
aa
bb
a
b
$\emptyset$

(f) (2 points) Let $X = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. Define an equivalence relation R on X by:

$$xRy \text{ if } x - y \text{ is divisible by } 3.$$

Then the partition $\mathcal{P}_R$ associated to the relation R is:

A. $\mathcal{P}_R = \{\{0, 3, 6, \dots\}, \{1, 4, 7, \dots\}, \{2, 5, 8, \dots\}\}$
B. $\mathcal{P}_R = \{\{0, 2, 4, \dots\}, \{1, 3, 5, \dots\}\}$
C. $\mathcal{P}_R = \{\{0\}, \{1\}, \{2\}, \dots\}$
D. $\mathcal{P}_R = \{(0,0), (0,3), (3,0) \dots, (3,3), (3,6), (6,3), \dots,$
$(1,1), (1,4), (4,1), \dots, (4,4), (4,7)(7,4), \dots,$
$(2,2), (2,5), (5,2), \dots, (5,5), (5,8)(8,5) \dots\}$

2. (6 points) Prove by induction that

$$\sum_{i=1}^{n}(2i-n)=n$$

$2(n+1) - n + 1$

$n+1$

for any integer $n \geq 1$.
Hint: Use

$$\sum_{i=1}^{n+1}(2i-(n+1)) = \left(\sum_{i=1}^{n+1}(2i-n)\right) - (n+1).$$

basis step:

$n=1 \quad \sum_{i=1}^{n} (2i-n) = 2(1)-(1) = 2-1=1$

$n = (1) = 1$

induction step:

Assume $\sum_{i=1}^{n} (2i-n)=n$ for some $n \geq 1$. Proof that it is true for $\sum_{i=1}^{n+1} (2i-(n+1))$:

$$\sum_{i=1}^{n+1} 2i-(n+1) = \left(\sum_{i=1}^{n+1} (2i-n)\right) - n+1$$

$$= \left(\sum_{i=1}^{n} 2i-n\right) + 2(n+1)-(1) - (n+1)$$

$$= n - n + 2(n+1) - (n+1)$$

$$= n+1$$

Hence, the statement is true for some $n+1$

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3. In the following questions, simply write down your answer. There is *no justification needed.* Do not simplify expressions as $2^4, 6!, C(n,r), \ldots$.

(a) (2 points) Let $X = \{1, 2, 3, 4, 5, 6\}$. Determine the number of elements of the following set

$$\{S \mid S \text{ is a subset of } X \text{ and } |S| = 4\}.$$

$n = 6$
$r = 4$

$\boxed{C(6,4)}$

(b) (2 points) Determine the number of 5-bit strings starting in 101.

$\boxed{101}\ \square\square$

$\boxed{2^2}$

(c) (2 points) Determine the number of 5-bit strings ending in 010.

$\square\square\ \boxed{010}$

$\boxed{2^2}$

(d) (2 points) Determine the number of 5-bit strings starting in 101 or ending in 010.

$\boxed{101}\ \square\square$ $\quad$ $\square\square\ \boxed{010}$ $\quad$ no overlap

$\boxed{2^2 + 2^2}$

(e) (2 points) You have three friends: Rocco, Gina and Hans. And you have seven different sweets: a popsicle, a piece of apple pie, a chocolate bar, a Berliner, a jelly doughnut, a marshmallow and a lemon drop.
You want to give two sweets to Rocco, three to Gina and two to Hans. In how many ways could you do this?

$n = 7$

$n_1 = 2$
$n_2 = 3$
$n_3 = 2$

$$\frac{n!}{n_1! n_2! n_3!} \to \boxed{\frac{7!}{2!3!2!}}$$

4. Answer the following questions, *justifying* your answers. (If an answer is Yes, explain why. If an answer is No, give a counterexample.)

(a) (2 points) Let $X = \{a, b, c, \ldots, z\}$ be the alphabet. Let α and β be strings over X. Is

$$\alpha\beta = \beta\alpha?$$

No $\alpha = az$

$\beta = cqx$

$\alpha\beta = azcqx$

$\beta\alpha = cqxaz$

$azcqx \neq cqxaz$ so $\alpha\beta \neq \beta\alpha$

(b) (2 points) Let A, B, C be sets. Then

$$A \cap (B \cup C) = (A \cap B) \cup C.$$

No, if

$A = \{1, 2\}$

$B = \{2, 4\}$

$C = \{1, 2, 3\}$

$(B \cup C) = \{1, 2, 3, 4\}$

$A \cap (B \cup C) = \{1, 2\}$

$(A \cap B) = \{2\}$

$(A \cap B) \cup C = \{1, 2, 3\}$

Hence $A \cap (B \cup C) \neq (A \cap B) \cup C$

(c) (2 points) Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$. Is there a bijective function $f : X \to Y$?

Yes, $|X| = |Y| = 4$, and each element in X is unique and each element in Y is unique, so every x has a y (inj) and every y has an x (sur) so there is a bij. function $f : X \to Y$

that doesn't really "mean anything". A set can't have one element twice

(d) (2 points) Define $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$. Is f surjective?

Yes, for every $x^2 \in \mathbb{R}_{\geq 0}$ there is a ~~unique~~ $x \in \mathbb{R}$, so f is surjective