Name: _

UCLA II, ..

Section letter: ___1A___

Math 61 : Discrete Structures
Midterm 2
Instructor: Ciprian Manolescu

You have 50 minutes.

No books, notes or calculators are allowed.
Do not use your own scratch paper.

1. (1 point each) True or False: Circle the right answers. You do NOT need to justify your answers.

There exists a graph with 100 vertices, all of degree 2.   **T** / F

There exists a graph with 100 vertices, all of degree 3.   **T** / F

The complete bipartite graph $K_{4,4}$ has an Euler cycle.   **T** / F

The complete bipartite graph $K_{4,4}$ has a Hamiltonian cycle.   **T** / F

In a graph, every simple cycle is a simple path.   T / **F**

There exist graphs with no cycles.   **T** / F

There exist graphs that have cycles, but no simple cycles.   T / **F**

The n-cube H_n is bipartite, for all $n \geq 1$.   **T** / F

There exists a graph that has an Euler cycle, and also an Euler path between different vertices.   T / **F**

For any n and k, we have $C(n,k) + C(n-1,k) = C(n,k+1)$.   T / **F**

If a sequence s_n satisfies the recurrence $s_n = s_{n-1} + s_{n-2}$, then s_n have to be the Fibonacci numbers.   T / **F**

Suppose we want to find the shortest path from a to z in a connected graph with positive weights. Then, by the end of Dijkstra's algorithm, there can be no vertices v labeled with $L(v) = \infty$.   T / **F**

2. (2 points each) **Multiple choice:** Circle the right answers. You do NOT need to justify your answers.
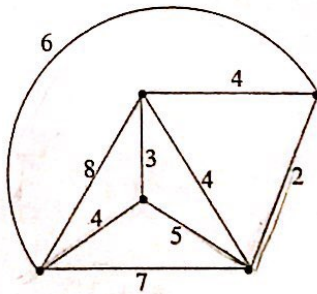
(a) The expression

$$\frac{C(n,k)}{C(n,k-1)}$$

can be simplified to

(A) $\dfrac{k}{k-1}$;   (B) k;   (C) $\dfrac{n-k+1}{k}$;   (D) $\dfrac{1}{k(n-k)}$;   (E) $\dfrac{k}{n-k}$.

(b) Consider the following weighted graph:

In the solution to the traveling salesman problem, the shortest Hamiltonian cycle has length

(A) 18;   (B) 19;   (C) 20;   (D) 21;   (E) 22

3. Write down the answer to the following questions. *You do NOT need to justify your answers.*

(a) (3 points) What is the coefficient of $\underline{x^4 y^5 z^6}$ in the expansion of $(x - y - 2z)^{15}$?

$$\frac{15!}{4!\,5!\,6!} \, x^4 (-y)^5 (-2z)^6.$$

$$= \frac{15!}{4!\,5!\,6!} (-1)^5 (-2)^6 \, x^4 y^5 z^6.$$

$$= 3104640 \; x^4 y^5 z^6$$

The coefficient of $x^4 y^5 z^6$ is $\boxed{\dfrac{15!}{4!\,5!\,6!} (-1)^5 (-2)^6 = -3104640}$.

3

(b) (3 points) The Catalan number C_n is defined to be the number of paths from the point $(0,0)$ to (n,n), made of steps of length 1 taken either rightward or upward, such that we are allowed to touch but not go above the diagonal line from $(0,0)$ to (n,n). Write down a recurrence relation for C_n.

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k} \qquad \checkmark \qquad 3$$

$$= \frac{C(2n, n)}{n+1}$$

$2X\sqrt{3}$ $\dfrac{\sqrt{3}}{9}$ $=\sqrt{3}$ $\dfrac{2}{4}\sqrt{3}$ $\dfrac{2}{4}\sqrt{3}$ $2\sqrt{3}\times\frac{1}{2}=\sqrt{3}$ $\dfrac{9}{4}\sqrt{3}$

$3\times\dfrac{2\sqrt{3}}{3}\times\frac{1}{2}$ $\dfrac{\sqrt{3}}{3}$ $3\times\dfrac{2\sqrt{3}}{3}\times\frac{1}{2}=\sqrt{3}$

$\dfrac{\sqrt{3}}{2}\times\frac{1}{2}$ $\dfrac{3}{2}$

4. (8 points) Twenty objects are placed in the interior of an equilateral triangle of side length 3. Suppose no three of these objects are on the same line. Prove that we can find three objects that are the vertices of a triangle of area at most $\sqrt{3}/4$. *Show all your work.*

$\dfrac{1\sqrt{3}}{2}$

$\dfrac{\sqrt{3}}{4}$

$3\times\dfrac{3\sqrt{3}}{2}\times\frac{1}{2}$

$\dfrac{9}{4}\sqrt{3}$

Let 20 objects be pigeons, and 9 equilateral triangle of side 1 are divided from the original triangle be holes.

By pigeonhole theorem, at least $\lceil\frac{20}{9}\rceil = 3$ objects will be placed in the same triangle.

~~Since no three vertices are on the same line, the largest triangle~~

The largest area of triangle formed by choosing 3 objects will be $\dfrac{\sqrt{3}}{4}$

with 3 objects on the vertices of the equilateral triangle with side 1.

$1\dfrac{\sqrt{3}}{2}\times\frac{1}{2}=\dfrac{\sqrt{3}}{4}$

5. (10 points) Solve the recurrence relation

$$a_n = a_{n-1} \cdot (a_{n-2})^6$$

with the initial conditions $a_0 = 9, a_1 = 3$. *Show all your work.* Hint: Consider $\log_3(a_n)$.

Let $b_n = \log_3 a_n$

$\log_3 3 = 1$

$b_n = \log_3 (a_{n-1} \cdot (a_{n-2})^6) = \log_3 a_{n-1} + 6 \log_3 a_{n-2}$

$$b_n = b_{n-1} + 6 b_{n-2}$$

$$t^2 - t - 6 = 0$$

$$(t-3)(t+2) = 0$$

$$r_1 = 3 \quad r_2 = -2$$

$$\Rightarrow b_n = b3^n + d(-2)^n$$

$a_0 = 9 \rightarrow b_0 = \log_3 9 = 2$

$$b + d = 2 \quad \rightarrow 2b + 2d = 4 \quad ①$$

$a_1 = 3 \rightarrow b_1 = \log_3 3 = 1$

$$3b - 2d = 1 \quad ②$$

$$① + ② \quad 5b = 5 \rightarrow b = 1$$

$$d = 1$$

$$\Rightarrow b_n = 3^n + (-2)^n$$

$$a_n = 3^{b_n} = 3^{3^n + (-2)^n}$$