1. (1 point each) **True or False:** Circle the right answers. You do NOT need to justify your answers.

There exists a graph with 100 vertices, all of degree 2. **T / F**

There exists a graph with 100 vertices, all of degree 3. **T / F**

The complete bipartite graph $K_{4,4}$ has an Euler cycle. both even ✓ **T / F**

The complete bipartite graph $K_{4,4}$ has a Hamiltonian cycle. m=n ✓ **T / F**

In a graph, every simple cycle is a simple path. no repeated vertex **T / F**

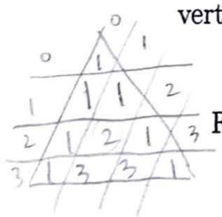
There exist graphs with no cycles. **T / F**

There exist graphs that have cycles, but no simple cycles. **T / F**

The n-cube H_n is bipartite, for all $n \geq 1$. **T / F**

$\delta(v) = \text{even } \forall v \in V$ $\quad\quad$ $\delta(v), \delta(w)$ odd

There exists a graph that has an Euler cycle, and also an Euler path between different vertices. **T / F**



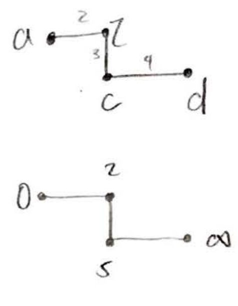
$C(n+1,k) = C(n,k) + C(n,k-1)$

For any n and k, we have $C(n,k) + C(n-1,k) = C(n,k+1)$. **T / F**

$C(n,k) = C(n-1,k) + C(n-1,k-1)$

If a sequence s_n satisfies the recurrence $s_n = s_{n-1} + s_{n-2}$, then s_n have to be the Fibonacci numbers. **T / F**

Suppose we want to find the shortest path from a to z in a connected graph with positive weights. Then, by the end of Dijkstra's algorithm, there can be no vertices v labeled with $L(v) = \infty$. **T / F**



2. (2 points each) **Multiple choice:** Circle the right answers. You do NOT need to justify your answers.

(a) The expression

$$\frac{C(n,k)}{C(n,k-1)}$$

$\frac{n(k-1)}{n-k+1}$ $\quad C(n,k) = \frac{n!}{k!\,(n-k)!}$

$C(n,k-1) = \frac{n!}{(n-k+1)!(k-1)!}$

can be simplified to

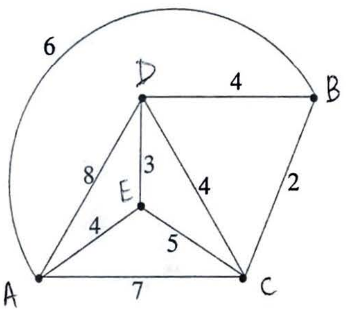
$(A)\ \frac{k}{k-1}$; $\quad (B)\ k$; $\quad (C)\ \frac{n-k+1}{k}$; $\quad (D)\ \frac{1}{k(n-k)}$; $\quad (E)\ \frac{k}{n-k}$.

$\frac{(n-k+1)!\,(k-1)!}{n!} \cdot \frac{\cancel{n!}}{k!\,(n-k)!}$

$\frac{n-k+1}{k}$

$\frac{(n-k+1)\cancel{(n-k)!}\,\cancel{(k-1)!}}{1} \cdot \frac{1}{k\cancel{(k-1)!}\,\cancel{(n-k)!}}$

(b) Consider the following weighted graph:



ABCDEA
6+2+4+3+4

ABDCEA
6+4+4+5

ABCEDA
6+2+5+3+8

A

~~ABDECA~~
~~6+4+3+5+7~~

In the solution to the traveling salesman problem, the shortest Hamiltonian cycle has length

(A) 18; $\quad (B)$ 19; $\quad (C)$ 20; $\quad (D)$ 21; $\quad (E)$ 22

3. Write down the answer to the following questions. *You do NOT need to justify your answers.*

a b c

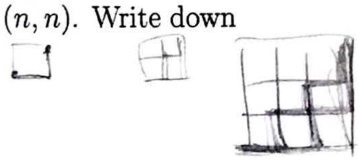
(a) (3 points) What is the coefficient of $x^4y^5z^6$ in the expansion of $(x - y - 2z)^{15}$?

$$\frac{n!}{i!j!k!} \qquad \frac{15!}{4!5!6!}(x)^4(-y)^5(-2z)^6 = -\frac{15!64}{4!5!6!}x^4y^5z^6$$

$$\boxed{-\frac{15!(64)}{4!5!6!}}$$

✓ 3

(b) (3 points) The Catalan number C_n is defined to be the number of paths from the point $(0,0)$ to (n,n), made of steps of length 1 taken either rightward or upward, such that we are allowed to touch but not go above the diagonal line from $(0,0)$ to (n,n). Write down a recurrence relation for C_n.



$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}$$

$$(n+2)C_{n+1} = (4n+2)C_n$$

$$C_{n+1} = \frac{4n+2}{n+2}C_n \quad \text{for } n \geq 1, \text{ initial conditions: } C_0 = 1,\ C_1 = 1$$

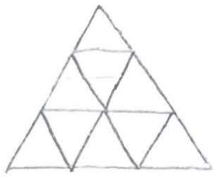
✓ 3

$$C_n = \frac{4(n-1)+2}{(n-1)+2}C_{n-1}$$

$$C_n = \frac{4n-4+2}{n+1}C_{n-1}$$

$$\boxed{C_n = \frac{4n-2}{n+1}C_{n-1} \quad \text{for } n \geq 1, \text{ initial conditions: } C_0 = 1}$$

4. (8 points) Twenty objects are placed in the interior of an equilateral triangle of side length 3. Suppose no three of these objects are on the same line. Prove that we can find three objects that are the vertices of a triangle of area at most $\sqrt{3}/4$. *Show all your work.*



Area of triangle $= \frac{1}{2}bh$

$= \frac{1}{2}(3)\left(\frac{3\sqrt{3}}{2}\right)$

$= \frac{9\sqrt{3}}{4}$

Split the triangle into 9 equilateral triangles w/ side length 1.

Each has area $A = \left(\frac{1}{9}\right)\left(\frac{9\sqrt{3}}{4}\right)$

$= \frac{\sqrt{3}}{4}$

Let the boxes = sub-triangles of side length 1

$n = \#$ boxes $= 9$

$m = \#$ objects $= 20$

$$\left\lceil \frac{m}{n} \right\rceil = \left\lceil \frac{20}{9} \right\rceil = \lceil 2.2... \rceil = 3$$

By the Pigeonhole Principle, there exists 1 sub-triangle of area $\frac{\sqrt{3}}{4}$ that contains at least 3 objects, so these 3 objects can form a triangle of area $\leq \frac{\sqrt{3}}{4}$.

5. (10 points) Solve the recurrence relation

$$a_n = a_{n-1} \cdot (a_{n-2})^6$$

with the initial conditions $a_0 = 9, a_1 = 3$. *Show all your work.* Hint: Consider $\log_3(a_n)$.

$\log_3(a_n) = \log_3(a_{n-1} \cdot (a_{n-2})^6)$

$\log_3(a_n) = \log_3(a_{n-1}) + 6\log_3(a_{n-2})$

let $b_n = \log_3(a_n)$

$b_n = b_{n-1} + 6b_{n-2}$ $\quad b_0 = 2,\ b_1 = 1$

$t^2 = t + 6$

$t^2 - t - 6 = 0$

$(t-3)(t+2) = 0$

$t = 3 \quad t = -2$

$b_n = x(3)^n + y(-2)^n$

$\begin{cases} 2 = x(3)^0 + y(-2)^0 \\ 1 = x(3) + y(-2) \end{cases}$

$x + y = 2$

$y = 2 - x$

$1 = 3x - 2y$

$1 = 3x - 2(2-x)$

$1 = 3x - 4 + 2x$

$5x = 5x$

$x = 1$

$y = 1$

$b_n = 3^n + (-2)^n$

$3^{b_n} = a_n$

So $\boxed{a_n = 3^{3^n + (-2)^n}}$

Do not write on this page.

1	11	out of 12 points
2	4	out of 4 points
3	6	out of 6 points
4	8	out of 8 points
5	10	out of 10 points
Total	39	out of 40 points