1. (1 point each) **True or False:** Circle the right answers. You do NOT need to justify your answers.

There exists a graph with 100 vertices, all of degree 2.  T / F

There exists a graph with 100 vertices, all of degree 3. T / F

The complete bipartite graph $K_{4,4}$ has an Euler cycle.  T / F

The complete bipartite graph $K_{4,4}$ has a Hamiltonian cycle. T / F

In a graph, every simple cycle is a simple path. T / F

There exist graphs with no cycles. repeated edge T / F

There exist graphs that have cycles, but no simple cycles. T / F

~~no~~ repeated vertices

The n-cube H_n is bipartite, for all $n \geq 1$. T / F

There exists a graph that has an Euler cycle, and also an Euler path between different vertices. T / F

For any n and k, we have $C(n,k) + C(n-1,k) = C(n,k+1)$. T / F

If a sequence s_n satisfies the recurrence $s_n = s_{n-1} + s_{n-2}$, then s_n have to be the Fibonacci numbers. initial conditions T / F

Suppose we want to find the shortest path from a to z in a connected graph with positive weights. Then, by the end of Dijkstra's algorithm, there can be no vertices v labeled with $L(v) = \infty$. T / F

2. (2 points each) **Multiple choice:** Circle the right answers. You do NOT need to justify your answers.

(a) The expression

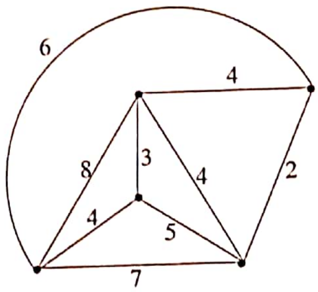
$$\frac{C(n,k)}{C(n,k-1)}$$

$$\frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k-1)!}} = \frac{(k-1)!(n-k-1)!}{k(k-1)!\cdot(n-k)\cdot(n-k-1)!} = \frac{1}{k(n-k)}$$

can be simplified to

(A) $\frac{k}{k-1}$; (B) k; (C) $\frac{n-k+1}{k}$; (D) $\frac{1}{k(n-k)}$; (E) $\frac{k}{n-k}$.

(b) Consider the following weighted graph:



$2+4+3+4+6+2=21$

$3+4+6+2+4$
$=19$

$2+4+3+4+6$
$=19$

$4+3+4+2+6$

In the solution to the traveling salesman problem, the shortest Hamiltonian cycle has length

(A) 18; (B) 19; (C) 20; (D) 21; (E) 22

3. Write down the answer to the following questions. *You do NOT need to justify your answers.*

(a) (3 points) What is the coefficient of $x^4y^5z^6$ in the expansion of $(x-y-2z)^{15}$?

$$\frac{n!}{4!5!6!} = \frac{15!}{4!5!6!} \cdot 1^4 \cdot (-1)^5 \cdot (-2)^6$$

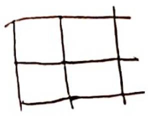
$$= (-2)^6(-1)^5 \cdot \frac{15!}{4!5!6!}$$

$$= 64 \cdot (-1) \dots$$

$$= \boxed{(-64) \cdot \frac{15!}{4!5!6!}}$$

3

(b) (3 points) The Catalan number C_n is defined to be the number of paths from the point $(0,0)$ to (n,n), made of steps of length 1 taken either rightward or upward, such that we are allowed to touch but not go above the diagonal line from $(0,0)$ to (n,n). Write down a recurrence relation for C_n.



$C_2 = 2$

3

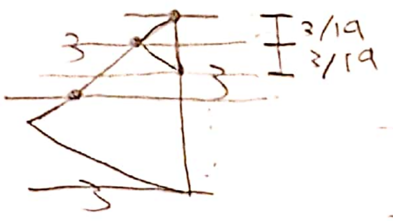
$$\boxed{C_n = \sum_{k=1}^{n} C_{n-k} \cdot C_{n-1}}$$

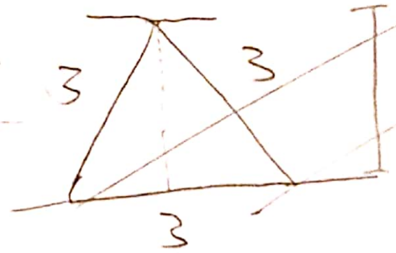
$C_1 \cdot C_0 + C_0 \cdot C_1$

$C_2 = 1 + 1 = 2$

$$\boxed{\begin{array}{l} C_0 = 1 \\ C_1 = 1 \end{array}}$$

4. (8 points) Twenty objects are placed in the interior of an equilateral triangle of side length 3. Suppose no three of these objects are on the same line. Prove that we can find three objects that are the vertices of a triangle of area at most $\sqrt{3}/4$. *Show all your work.*





$3^2 = 1.5^2 + h^2$

$h^2 = -(1.5)^2 + 3^2$

$h^2 = 9 - 2.25$

$h^2 = 6.75$

$h =$

space the lines as much as possible starting from a vertex and ending at a vertex.

space between each line: $\frac{3}{19}$

alternate placing objects on vertical side of Δ and on diagonal side of Δ to maximize distance

smallest Δ: top triangle near vertex

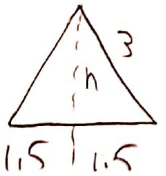
$L \cdot H$

$\left(\frac{1}{2}\right)\left(6/19\right) \cdot \left(6/19 \cdot \text{height}\right)$ of overall Δ

$h = \sqrt{9 - 1.5^2}$

$= \sqrt{9 - 2.25}$

$h = \sqrt{6.75}$



area smallest $\Delta = \frac{1}{2}\left(\frac{6}{19}\right)\left(\frac{6}{19}\right)\left(\sqrt{6.75}\right)$

$= \frac{18}{(19)^2}\left(\sqrt{6.75}\right)$

$\frac{\sqrt{3}}{4} > \frac{18}{(19)^2}\left(\sqrt{6.75}\right)$

$a_2 = 3 \cdot (9)^6$
$= 3 \cdot 3^{12} = 3^{13}$

$\log_3(3^6)$ $\log_3(3)^6$
$= 6$ $= 6$

5. (10 points) Solve the recurrence relation

$$a_n = a_{n-1} \cdot (a_{n-2})^6$$

with the initial conditions $a_0 = 9, a_1 = 3$. *Show all your work.* Hint: Consider $\log_3(a_n)$.

$b_n = \log_3(a_n)$ $a_n = 3^{b_n}$ $b_0 = \log_3(9)$ $b_1 = \log_3(3)$
$b_0 = 2$ $b_1 = 1$

$\log_3(a_n) = \log(a_{n-1} \cdot (a_{n-2})^6)$

$\log_3(a_n) = \log_3(a_{n-1}) + \log_3((a_{n-2})^6)$

$\log_3(a_n) = \log_3(a_{n-1}) + 6\log_3(a_{n-2})$

$b_n = b_{n-1} + 6b_{n-2}$

$t^2 = c_1 t + c_2$

$t^2 - t - 6 = 0$

$(t-3)(t+2) = 0$

$t = 3, -2$

$r_1 = 3$ $r_2 = -2$

$b_n = b \cdot r_1^n + d \cdot r_2^n = b \cdot (3)^n + d \cdot (-2)^n$

$b_0 = 2 = b \cdot 3^0 + d \cdot (-2)^0$

$2 = b + d \rightarrow b = 2 - d$

$b_1 = 1 = b \cdot 3^1 + d(-2)^1$

$1 = 3b - 2d$ $1 = 3(2-d) - 2d$

$1 = 6 - 3d - 2d$ $-5 = -5d$

$d = 1$ $b = 2 - 1$

$b = 1$

$b_n = 3^n + (-2)^n$ $a_n = 3^{b_n}$

$\boxed{a_n = 3^{3^n + (-2)^n}}$ $a_2 = 3^{9+4} = 3^{13}$