Math 61 : Discrete Structures
Midterm 2
Instructor: Ciprian Manolescu

You have 50 minutes.

No books, notes or calculators are allowed.
Do not use your own scratch paper.

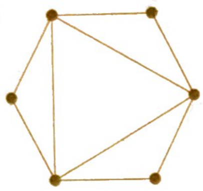
9

1. (10 points) **True or False:** Circle the right answers. You do NOT need to justify your answers.

Consider the following graphs:

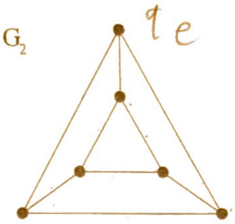
6 v
9e

G_1



6 v
9e

G_2



The incidence matrices of G_1 and G_2 have the same number of rows. T / F ✓

vertices x rows

The incidence matrices of G_1 and G_2 have the same number of columns. T / F ✓

G_1 and G_2 are isomorphic. T / F ✓

G_1 and G_2 both have Hamiltonian cycles. T / F

G_1 admits an Euler cycle. T / F

G_2 admits an Euler cycle. T / F

G_1 admits an Euler path between different vertices. T / F ✓

G_2 admits an Euler path between different vertices. T / F ✓

Unrelated to the picture above:

There exists a graph with six vertices, of degrees $4, 4, 4, 4, 4, 5$. T / F ✓

There exists a graph with six vertices, of degrees $4, 4, 4, 4, 5, 5$. T / F ✓

2. (6 points) Write down the answer to the following questions. *You do NOT need to justify your answers.*

If a graph has 100 vertices, all of degree 3, then how many edges does it have?

sum of vertice degrees = 2 • edges

300 = 2e

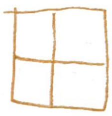
$\boxed{150 \text{ edges}}$

3

Let d_n be the number of ways one can fill a $2 \times n$ rectangle with 2×1 dominoes. Write down a recurrence relation for d_n.

d_0, 1 way, not at all

d_1 =  1 way

d_2 =  2

d_3 = 

3

d_4 = 

5

3

$$\boxed{\begin{array}{l} d_n = d_{n-1} + d_{n-2} \\ d_0 = 1,\ d_1 = 1 \end{array}}$$

$\sqrt{16}$

3. (8 points) Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2}$$

with the initial conditions $a_0 = 2, a_1 = 2$. *Show all your work.*

$t^2 = 2t + 1 \qquad t^2 - 2t - 1 = 0$

$4 \cdot -1 \cdot 1 \quad 4$

$$t = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{4 \cdot 2}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Gen solution: $a_n = b r_1^n + d r_2^n \qquad t = 1+\sqrt{2},\ 1-\sqrt{2}$

$a_0 = b(1+\sqrt{2})^0 + d(1-\sqrt{2})^0 = 2 \qquad a_0 = b + d = 2 \quad b = 2 - d$

$a_1 = b(1+\sqrt{2}) + d(1-\sqrt{2}) = 2 \qquad a_1 = 2 - \cancel{d} + 2\sqrt{2} - \sqrt{2}d + \cancel{d} - \sqrt{2}d$

$a_1 = b + \sqrt{2}b + d - \sqrt{2}d = 2 \qquad 2 = 2 + 2\sqrt{2} - 2\sqrt{2}d$

$a_1 = 2 - d + \sqrt{2}(2-d) + d - \sqrt{2}d \qquad -2 \quad -2$

$$\frac{0}{2\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2}} - \frac{2\sqrt{2}d}{2\sqrt{2}}$$

$0 = 1 - d \qquad d = 1$

$b = 2 - d$

$b = 2 - 1$

$b = 1$

$$\boxed{a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n}$$

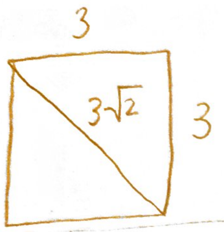
$\frac{9}{4}+\frac{9}{4}=\frac{18}{4}$ $1.5^2+1.5^2$ $(\frac{3}{2})^2+(\frac{3}{2})^2$

15×15: 75, 150, 2.25

$2.25 + 2.25 = 4.50$

4. (8 points) Ten objects are placed on a square of side length 3 feet. Prove that there exist two objects at distance at most $\sqrt{2}$ feet from each other. *Show all your work.*

Pigeonhole / Contradiction



$\frac{3\sqrt{2}}{2}$ $\frac{3\sqrt{2}}{4}$

$a^2+b^2=c^2$

$9+9=18$

$\sqrt{18}$

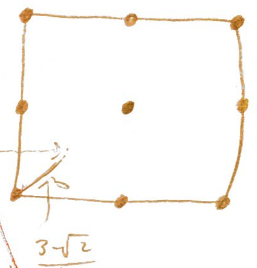
$\frac{\sqrt{4.50}}{2}$

Let us place nine objects on the square.

- One at each corner
- One in the center of each edge
- One in the center

Why is this "best"?

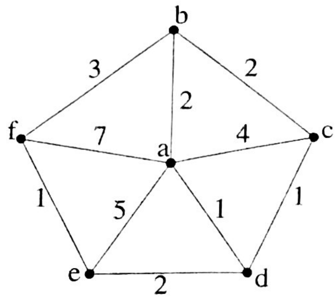
Like so:



At this point, all 9 objects are at a distance greater than $\sqrt{2}$ away from each other. These 9 "pigeons" fit into the 9 "pigeonholes." However, introducing a tenth object anywhere on that square would put it within $\sqrt{2}$ of any other already-present object. Even placing the tenth object as far as possible from all other objects would leave the closest one at $\frac{3-\sqrt{2}}{4}$, which is definitely less than $\sqrt{2}$.

3

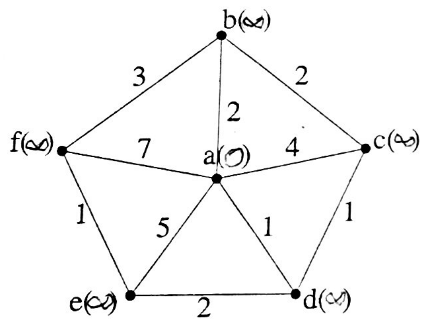
5. (8 points) Consider the following weighted graph:



Suppose you want to use Dijkstra's algorithm to find the length of the shortest path from a to f. Write down the initialization and the first two iterations of the algorithm. (You do NOT have to complete the whole algorithm.) At each step, write down if f is in T, what is the current node, what is the new set of unvisited vertices T, then which nodes change labels and how.

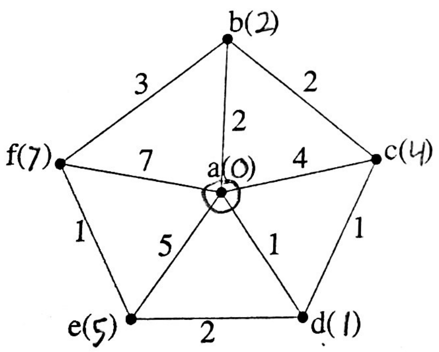
Also, please circle the visited labels at each step (the ones not in T), and write down the label at each vertex in paranthesis.

Initialization:



T= $\{a, b, c, d, e, f\}$

First iteration:



Is f in T? Yes

current node: a

T= $\{b, c, d, e, f\}$

changes in labels:

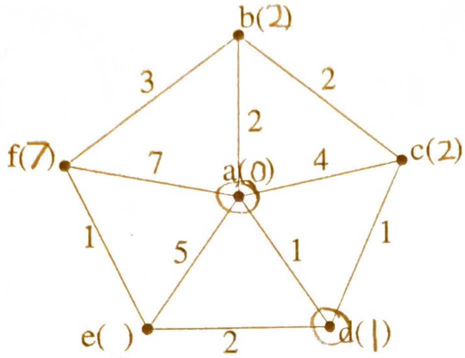
$b = \min(\infty, 0+2) = 2$
$c = \min(\infty, 0+4) = 4$
$d = \min(\infty, 0+1) = 1$
$e = \min(\infty, 0+5) = 5$
$f = \min(\infty, 0+7) = 7$

(Change only unvisited adjacent

(go to the next page)

Second iteration:

ok



Is f in T? Yes

current node: d

T= $\{b, c, e, f\}$

changes in labels: $c = \min(4, 1+1) = 2$

$e = \min(5, 1+2) = 3$

(only change unvisited adjacent)