Name: ______

UCLA ID Nur 30

Section letter: 1C

Math 61 : Discrete Structures
Midterm 2
Instructor: Ciprian Manolescu

You have 50 minutes.

No books, notes or calculators are allowed.
Do not use your own scratch paper.

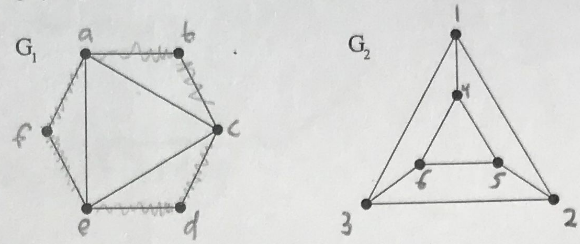
9

1. (10 points) **True or False:** Circle the right answers. You do NOT need to justify your answers.

Consider the following graphs:

G_1	
vertex	degree
a	4
b	2
c	4
d	2
e	4
f	2

G_2	
vertex	degree
1	3
2	3
3	3
4	3
5	3
6	3



$$\begin{array}{c} \\ v_1 \\ \vdots \\ v_n \end{array}\begin{array}{c} e_1 \cdots e_n \\ \left(\begin{array}{c} \\ \\ \\ \end{array} \right) \end{array}$$

The incidence matrices of G_1 and G_2 have the same number of rows. T / F

rows = # of vertices

G_1 = 6 vertices G_2 = 6 vertices

The incidence matrices of G_1 and G_2 have the same number of columns. T / F

columns = # of edges

G_1 = 9 edges G_2 = 9 edges

G_1 and G_2 are isomorphic. T / F

no - degrees don't match

To have an Euler cycle must:
1. connected
2. all vertices of even degree

G_1 and G_2 both have Hamiltonian cycles. T / F

False - G_1 doesn't have one - cuts off ecrly

G_1 admits an Euler cycle. T / F

true - connected, all vertices of even degree

G_2 admits an Euler cycle. T / F

no - not all vertices of even degree

Euler Path
- degree of v and w is odd
- all other degrees are even

G_1 admits an Euler path between different vertices. T / F

no - must have 2 vertices of odd degree

G_2 admits an Euler path between different vertices. T / F

no - must have vertices of even degree

Unrelated to the picture above:

There exists a graph with six vertices, of degrees 4, 4, 4, 4, 4, 5. T / F

4×5 = 20

20+5

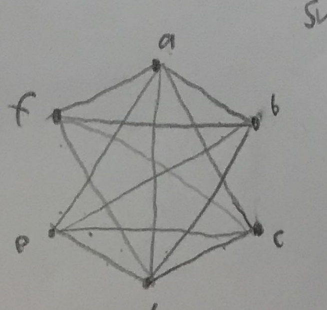
sum of degrees = 25 → must be even

There exists a graph with six vertices, of degrees 4, 4, 4, 4, 5, 5. T / F

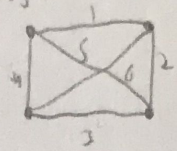
16 10

Sum of degrees = 26 ✓

a	5
b	4
c	4
d	5
e	4
f	4



$\frac{100 \cdot 3}{2} = 50 \cdot 3 = 150$ $\qquad$ $\frac{4 \times 3}{2} = \frac{12}{2} = 6$ $\qquad$ 4 vertices, all of degree 3 $\qquad$ $4 \times 3 = 12$, $\frac{12}{2} = 6$



2. (6 points) Write down the answer to the following questions. *You do NOT need to justify your answers.*

If a graph has 100 vertices, all of degree 3, then how many edges does it have?

100 vertices, each with degree 3

total degree = 300

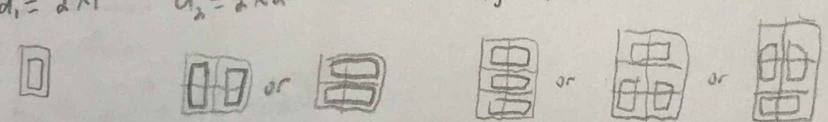
$\frac{300}{2} = \boxed{150 \text{ edges}}$

3

Let d_n be the number of ways one can fill a $2 \times n$ rectangle with 2×1 dominoes. Write down a recurrence relation for d_n.

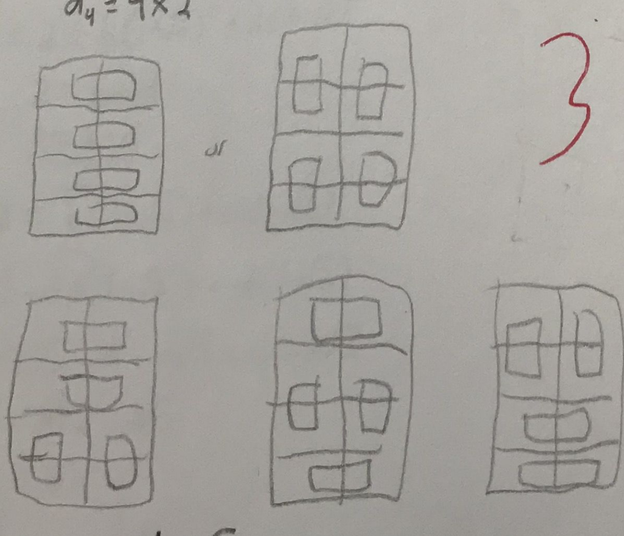
$S_{n-1} + S_{n-2}$

$d_1 = 2 \times 1$ $\qquad$ $d_2 = 2 \times 2$ $\qquad$ $d_3 = 3 \times 2$ $\qquad$ 1 2 3 5



$d_1 = 1$ $\qquad$ $d_2 = 2$ $\qquad$ $d_3 = 3$

$d_4 = 4 \times 2$



$d_4 = 5$

3

$$\boxed{\begin{aligned} d_n &= d_{n-1} + d_{n-2}, \quad n \geq 3 \\ d_1 &= 1 \\ d_2 &= 2 \end{aligned}}$$

3. (8 points) Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2}$$

with the initial conditions $a_0 = 2, a_1 = 2$. *Show all your work.*

Guess solution of the form: $V_n = t^n$

$$\frac{t^n}{t^{n-2}} = \frac{2t^{n-1}}{t^{n-2}} + \frac{t^{n-2}}{t^{n-2}}$$

$$t^2 = 2t + 1$$

$$t^2 - 2t - 1 = 0$$

$$t = \frac{2 \pm \sqrt{4-4(1)(-1)}}{2} = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm \sqrt{4}\sqrt{2}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = \frac{\cancel{2}(1 \pm \sqrt{2})}{\cancel{2}} = 1 \pm \sqrt{2}$$

$$t = 1 \pm \sqrt{2}$$

$$V_n = (1+\sqrt{2})^n \quad W_n = (1-\sqrt{2})^n$$

General Solution

$$a_n = bV_n + dW_n$$

$$a_n = b(1+\sqrt{2})^n + d(1-\sqrt{2})^n$$

Base Cases

$n=0$

$$a_0 = 2 = b(1+\sqrt{2})^0 + d(1-\sqrt{2})^0$$

$$2 = b + d$$

$$b = 2 - d$$

$n=1$

$$a_1 = 2 = b(1+\sqrt{2}) + d(1-\sqrt{2})$$

$$2 = (2-d)(1+\sqrt{2}) + d(1-\sqrt{2})$$

$$\cancel{2} = \cancel{2} + 2\sqrt{2} - \cancel{d} - d\sqrt{2} + \cancel{d} - d\sqrt{2}$$

$$0 = 2\sqrt{2} - d\sqrt{2} - d\sqrt{2}$$

$$0 = 2\sqrt{2} - 2d\sqrt{2}$$

$$d\,2\sqrt{2} = 2\sqrt{2}$$

$$d = 1 \quad b = 1$$

$$\boxed{a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n}$$

$\frac{1.11}{9)10}$
$\frac{-9}{10}$

they can be in the interior

10 objects in 9 ft of space

$\lceil \frac{10}{9} \rceil = \lceil 1.111 \rceil = 2$

Yes.

n objects in k space

4. (8 points) Ten objects are placed on a square of side length 3 feet. Prove that there exist two objects at distance at most $\sqrt{2}$ feet from each other. *Show all your work.*

$\leq$ 1.4

~~Area = 9 ft~~

~~Area =~~

~~3~~

~~3~~

~~ways to place 10 objects - assume all are $\sqrt{2}$ apart~~

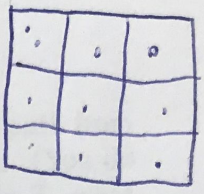
~~$a_1 \ldots a_{10}$~~

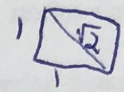
~~$(a_1 + \sqrt{2}) \ldots (a_{10} + \sqrt{2})$ = 20 items, $\sqrt{2}$ apart~~

~~20 items, $\sqrt{2}$ apart~~

~~would need $20 \times \sqrt{2} = 20\sqrt{2}$ ft^2 of space~~

~~However, our square only has 9 ft space,~~





ways to place objects - assuming exactly $\sqrt{2}$ ft apart

X = all possible positions

divide X into A and B

A: $a_1 \ldots a_{10}$

B: $(a_1 + \sqrt{2}$ ft$) \ldots (a_{10} + \sqrt{2}$ ft$)$

$|X| = 20$ items

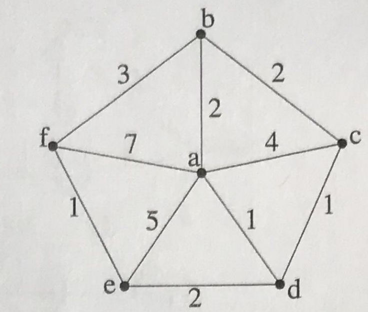
I don't understand.

For each item in X to be $\sqrt{2}$ ft apart, we would need $20 \times \sqrt{2} = 20\sqrt{2}$ ft^2 of space. However, our square only has $3 \times 3 = 9$ ft^2 of space. Therefore, by the Pigeonhole Principle, if we want to put 10 items, normally requiring $20\sqrt{2}$ ft^2 of space to be $\sqrt{2}$ apart, into only 9 ft^2 of space, then at least 2 objects at a distance less than or equal to $\sqrt{2}$ ft from each other.

1/8

$a \to f$

5. (8 points) Consider the following weighted graph:



Suppose you want to use Dijkstra's algorithm to find the length of the shortest path from $\boxed{a \text{ to } f}$. Write down the initialization and the first two iterations of the algorithm. (You do NOT have to complete the whole algorithm.) At each step, write down if f is in T, what is the current node, what is the new set of unvisited vertices T, then which nodes change labels and how.

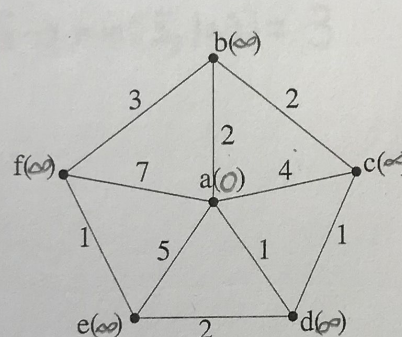
Also, please circle the visited labels at each step (the ones not in T), and write down the label at each vertex in paranthesis.

Initialization:

0 ∞

all in T

2/2

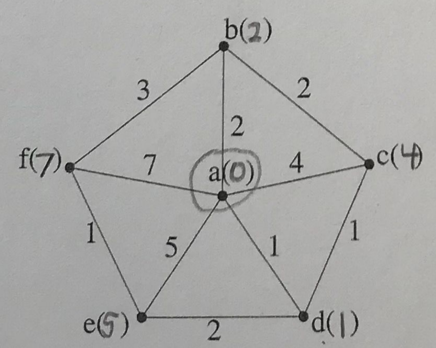


T= $\{a, b, c, d, e, f\}$

First iteration:

Circle a

3/3



Is f in T? Yes

current node: a

T= $\{b, c, d, e, f\}$

changes in labels: $b: \infty \to \min(\infty, 0+2) = 2$

(go to the next page)

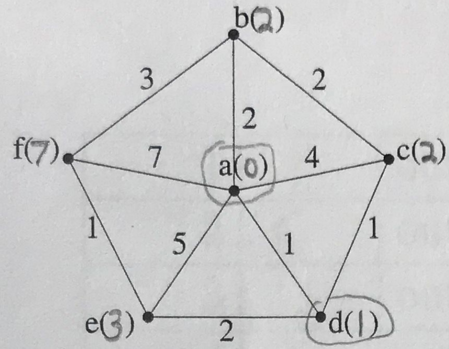
$c: \infty \to \min(\infty, 0+4) = 4$

$d: \infty \to \min(\infty, 0+1) = 1$

$e: \infty \to \min(\infty, 0+5) = 5$

$f: \infty \to \min(\infty, 0+7) = 7$

Second iteration:



3/3

Is f in T? Yes

current node: d

T= $\{b, c, e, f\}$

changes in labels:

$c: 4 \to \min(4, 1+1) = 2$

$e: 5 \to \min(5, 1+2) = 3$

Do not write on this page.

1	9	out of 10 points
2	6	out of 6 points
3	8	out of 8 points
4	1	out of 8 points
5	8	out of 8 points
Total	32	out of 40 points