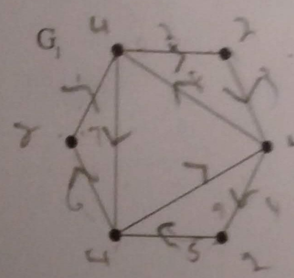
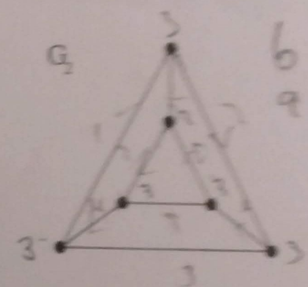
10

1. (10 points) **True or False:** Circle the right answers. You do NOT need to justify your answers.

Consider the following graphs:

6 ver.
9 edg.
Ham ✓





6 ver.
9 edg.
Ham ✓

The incidence matrices of G_1 and G_2 have the same number of rows. T / F

\# vertices

The incidence matrices of G_1 and G_2 have the same number of columns. T / F

\# edges

G_1 and G_2 are isomorphic. Different degrees of edges T / F

G_1 and G_2 both have Hamiltonian cycles. T / F

G_1 admits an Euler cycle. T / F

G_2 admits an Euler cycle. T / F

G_1 admits an Euler path between different vertices. T / F

G_2 admits an Euler path between different vertices. T / F

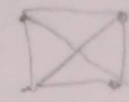
Unrelated to the picture above:

4+4+4+4+4+5 = 25

There exists a graph with six vertices, of degrees 4, 4, 4, 4, 4, 5. T / F

There exists a graph with six vertices, of degrees 4, 4, 4, 4, 5, 5. T / F

4+4+4+4+5+5 = 26



150

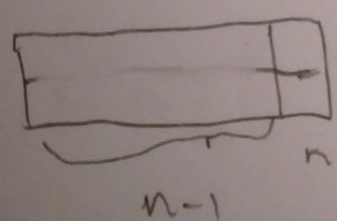
2. (6 points) Write down the answer to the following questions. *You do NOT need to justify your answers.*

$\frac{3 \cdot 100}{2} = 150$

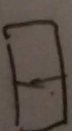
If a graph has 100 vertices, all of degree 3, then how many edges does it have?

$\boxed{150}$

Let d_n be the number of ways one can fill a $2 \times n$ rectangle with 2×1 dominoes. Write down a recurrence relation for d_n.

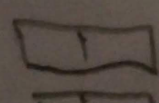


case 1

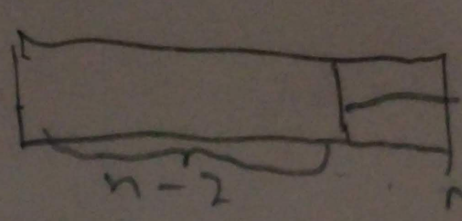
fill last place w/ . This takes up one spot, one way

d_{n-1}

case 2

fill last 2 slots w/ . One way to do this

d_{n-2}



$\boxed{d_{n-1} + d_{n-2}}$

n	a_n
2	6
3	14
4	34

3. (8 points) Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2}$$

with the initial conditions $a_0 = 2, a_1 = 2$. *Show all your work.*

$t^2 - 2t - 1 = 0$

$\frac{2 \pm \sqrt{4-4(-1)}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

Gen sol: $a_n = b(1+\sqrt{2})^n + d(1-\sqrt{2})^n$

Init cond.

$1+\sqrt{2}+1-\sqrt{2} = 2$

$\begin{cases} 2 = b + d \\ 2 = b(1+\sqrt{2}) + d(1-\sqrt{2}) \end{cases}$

$\begin{cases} 2(1+\sqrt{2}) = b(1+\sqrt{2}) + d(1+\sqrt{2}) \\ 2 = b(1+\sqrt{2}) + d(1-\sqrt{2}) \end{cases}$

$2+2\sqrt{2}-2 = d(1+\sqrt{2}) - d(1-\sqrt{2})$

$2\sqrt{2} = d(\cancel{1}+\sqrt{2}-\cancel{1}+\sqrt{2})$

$2\sqrt{2} = d(2\sqrt{2})$

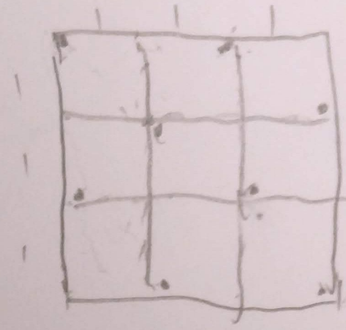
$d = 1$

$b = 1$

$1+\cancel{2\sqrt{2}}+2+(1-2\sqrt{2}+2)$

$\boxed{a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n}$

4. (8 points) Ten objects are placed on a square of side length 3 feet. Prove that there exist two objects at distance at most $\sqrt{2}$ feet from each other. *Show all your work.*

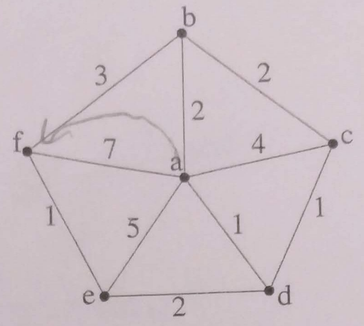


We can put 9 dots in different squares s.t. they are all $\geq \sqrt{2}$ feet apart

Let us first divide the square into 9 smaller squares each with side length 1 foot. Let the pigeonholes be the smaller squares. Then, the largest distance that we can put objects is if we can put them in different squares. Then, they could be $\geq \sqrt{2}$ feet apart (diagonal length of a small square). However, we have 10 objects (pigeons) that we want to put into 9 squares (pigeonholes). Therefore, there exists some square that will contain 2 objects, meaning those objects will be $< \sqrt{2}$ feet apart, or that object will $< \sqrt{2}$ feet apart from an object in another square by the pigeonhole principle

8

5. (8 points) Consider the following weighted graph:

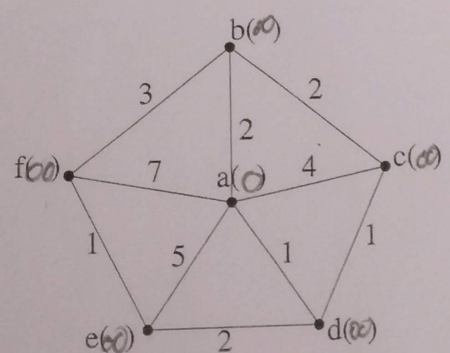


Suppose you want to use Dijkstra's algorithm to find the length of the shortest path from a to f. Write down the initialization and the first two iterations of the algorithm. (You do NOT have to complete the whole algorithm.) At each step, write down if f is in T, what is the current node, what is the new set of unvisited vertices T, then which nodes change labels and how.

Also, please circle the visited labels at each step (the ones not in T), and write down the label at each vertex in paranthesis.

Initialization:

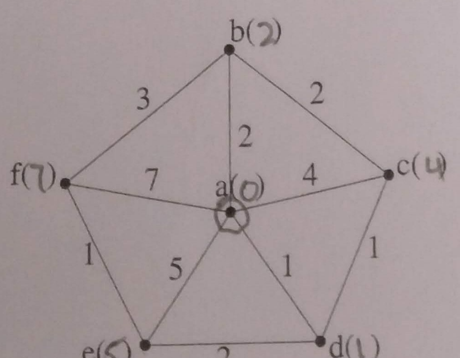
2/2



T= {a, b, c, d, e, f}

First iteration:

3/3



Is f in T? yes

current node: a

T= {b, c, d, e, f}

changes in labels:

b: ∞ → 2

c: ∞ → 4

d: ∞ → 1

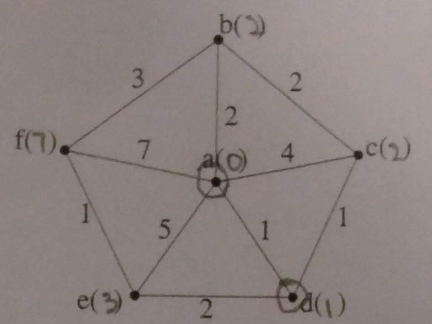
e: ∞ → 5

f: ∞ → 7

(go to the next page)

3/3

Second iteration:



Is f in T? yes

current node: d

T= {b, c, e, f}

changes in labels: e: 5 → 3

c: 4 → 2