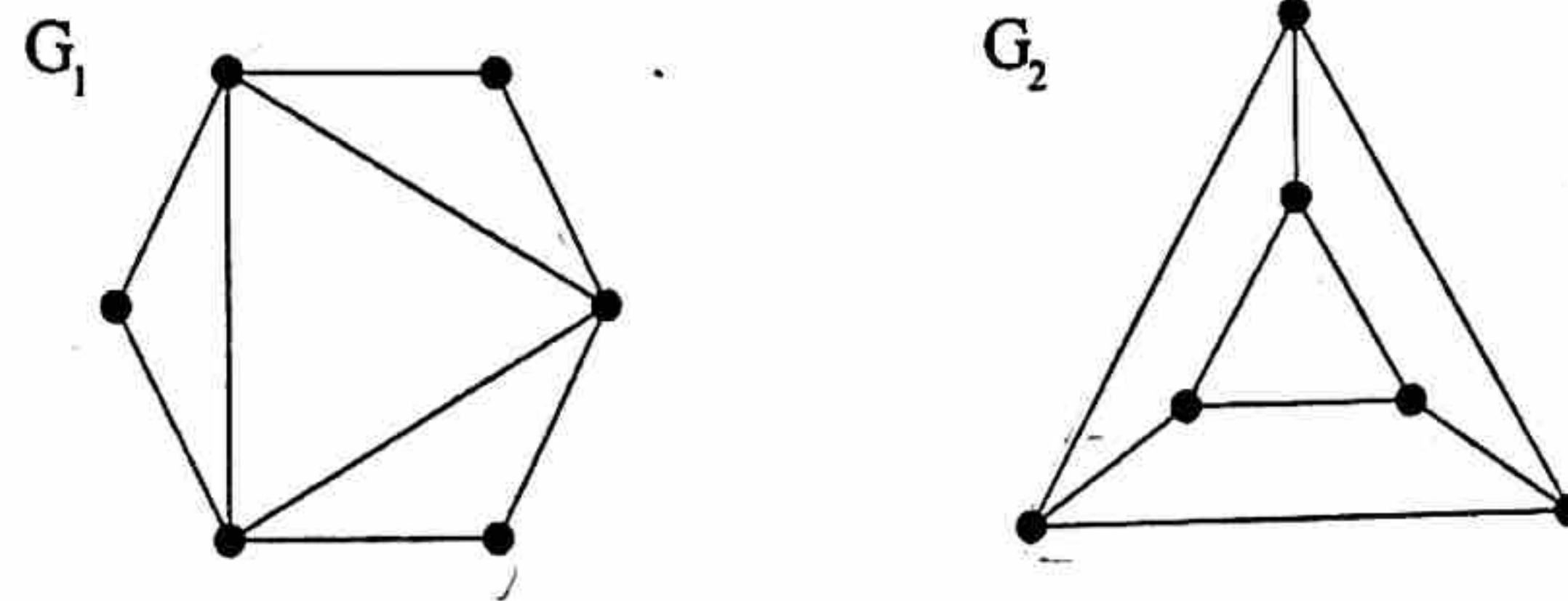


1. (10 points) **True or False:** Circle the right answers. You do NOT need to justify your answers.

Consider the following graphs:



The incidence matrices of G_1 and G_2 have the same number of rows.

T / F

The incidence matrices of G_1 and G_2 have the same number of columns.

T / F

G_1 and G_2 are isomorphic.

T / F

G_1 and G_2 both have Hamiltonian cycles.

T / F

G_1 admits an Euler cycle.

T / F

G_2 admits an Euler cycle.

T / F

G_1 admits an Euler path between different vertices.

T / F

G_2 admits an Euler path between different vertices.

T / F

Unrelated to the picture above:

There exists a graph with six vertices, of degrees 4, 4, 4, 4, 4, 5.

T / F

There exists a graph with six vertices, of degrees 4, 4, 4, 4, 5, 5.

T / F

2. (6 points) Write down the answer to the following questions. *You do NOT need to justify your answers.*

If a graph has 100 vertices, all of degree 3, then how many edges does it have?

150



Let d_n be the number of ways one can fill a $2 \times n$ rectangle with 2×1 dominoes. Write down a recurrence relation for d_n .

$$d_n = d_{n-1} + d_{n-2}$$



3. (8 points) Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2}$$

with the initial conditions $a_0 = 2, a_1 = 2$. Show all your work.

$$\text{Let } a_n = t^n$$

$$t^n = 2t^{n-1} + t^{n-2}$$

$$t^2 - 2t - 1 = 0$$

$$t = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

$$\therefore a_n = b(1+\sqrt{2})^n + d(1-\sqrt{2})^n$$

$$\begin{cases} 2 = b+d \\ 2 = b(1+\sqrt{2}) + d(1-\sqrt{2}) \end{cases}$$

$$\Rightarrow \begin{cases} 2 = b+d \\ 2 = b+d + \sqrt{2}(b-d) \end{cases}$$

$$\Rightarrow \begin{cases} b = d \\ 2 = b+d \end{cases}$$

$$\therefore b = d = 1$$

$$\therefore a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n \quad \checkmark$$

4. (8 points) Ten objects are placed on a square of side length 3 feet. Prove that there exist two objects at distance at most $\sqrt{2}$ feet from each other. Show all your work.

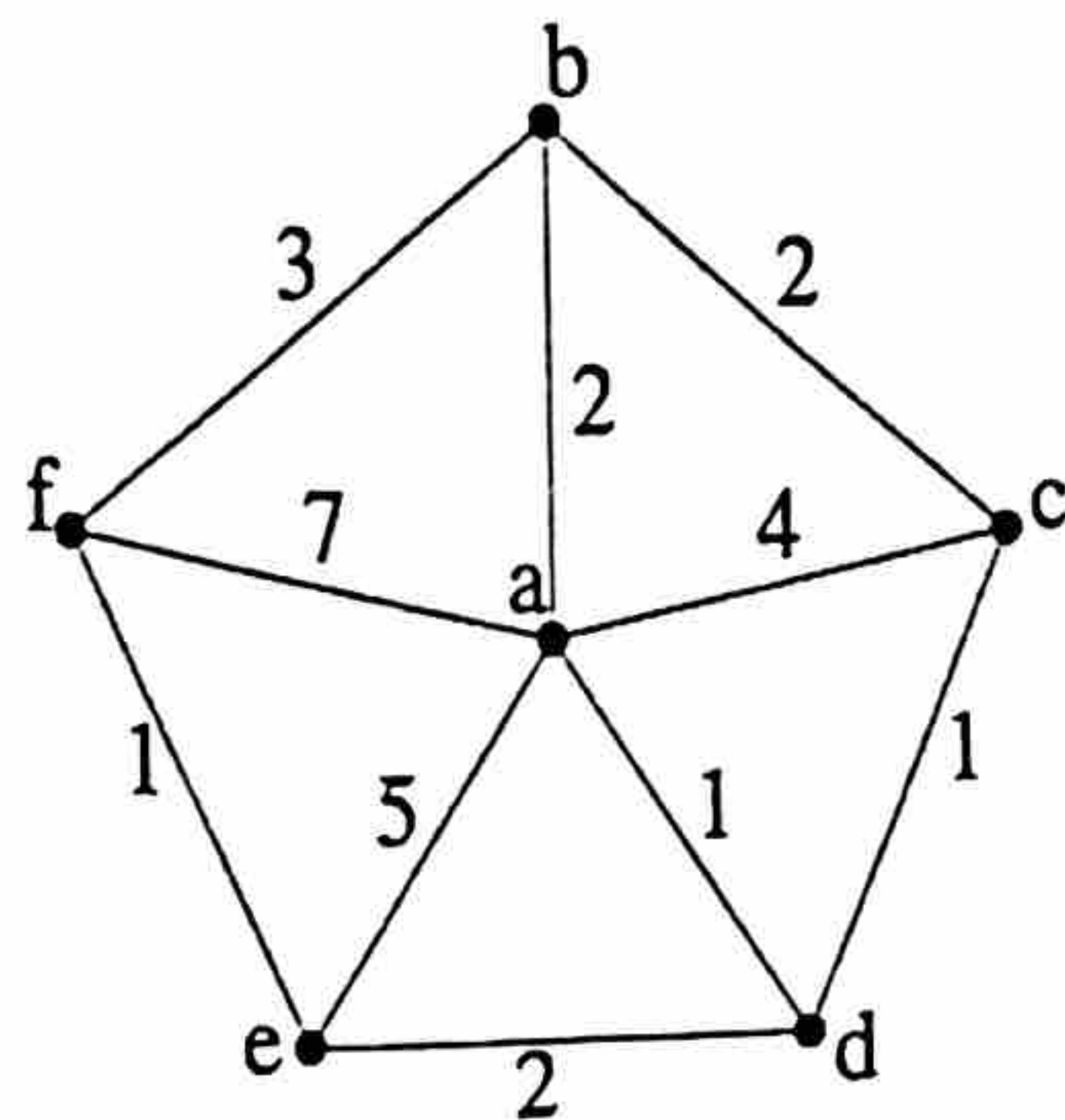
Split the square into 9 squares of side length 1

\therefore Within each square, the longest distance is $\sqrt{2}$.

Let the 9 squares be holes and 10 objects be pigeons. we are guaranteed to put 2 objects in the same square, and the 2 objects are at most $\sqrt{2}$ feet apart. (By pigeonhole principle)

8

5. (8 points) Consider the following weighted graph:

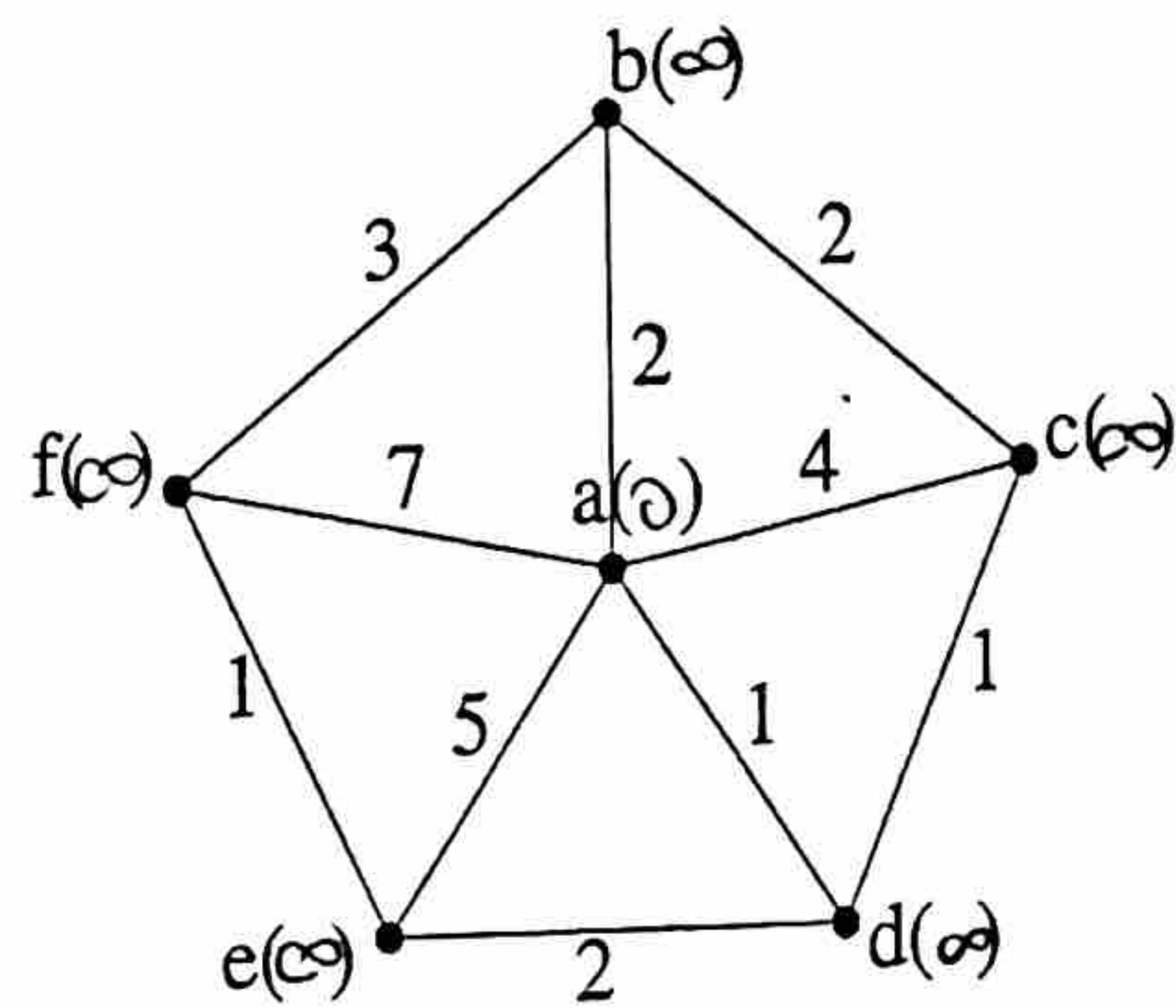


Suppose you want to use Dijkstra's algorithm to find the length of the shortest path from a to f . Write down the initialization and the first two iterations of the algorithm. (You do NOT have to complete the whole algorithm.) At each step, write down if f is in T , what is the current node, what is the new set of unvisited vertices T , then which nodes change labels and how.

Also, please circle the visited labels at each step (the ones not in T), and write down the label at each vertex in parenthesis.

Initialization:

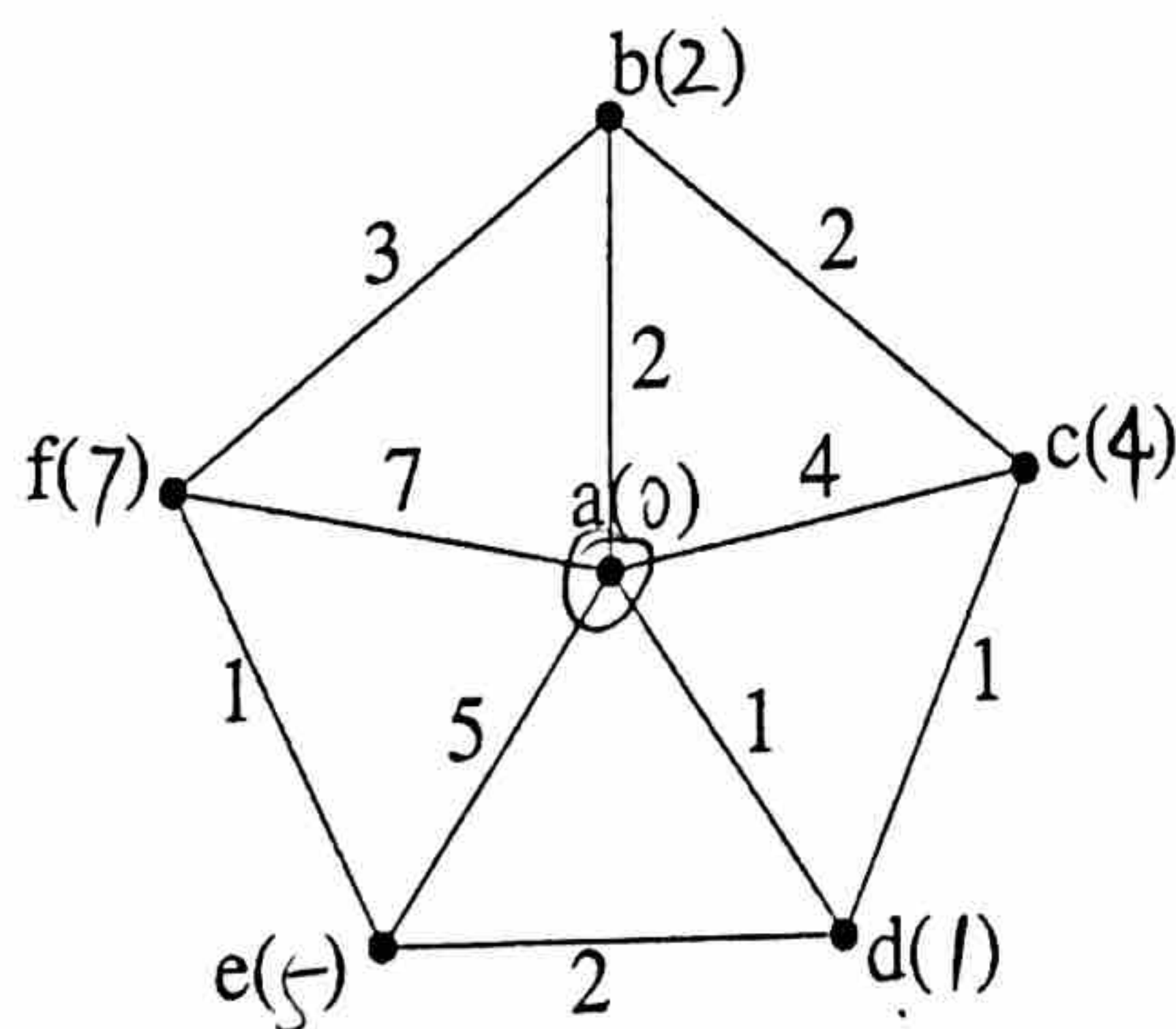
2/2



$$T = \{a, b, c, d, e, f\}$$

First iteration:

3/3



Is f in T ? Yes

current node: a

$$T = \{b, c, d, e, f\}$$

changes in labels: $L(b) = \min\{0+2, \infty\} = 2$

$$L(c) = \min\{0+4, \infty\} = 4$$

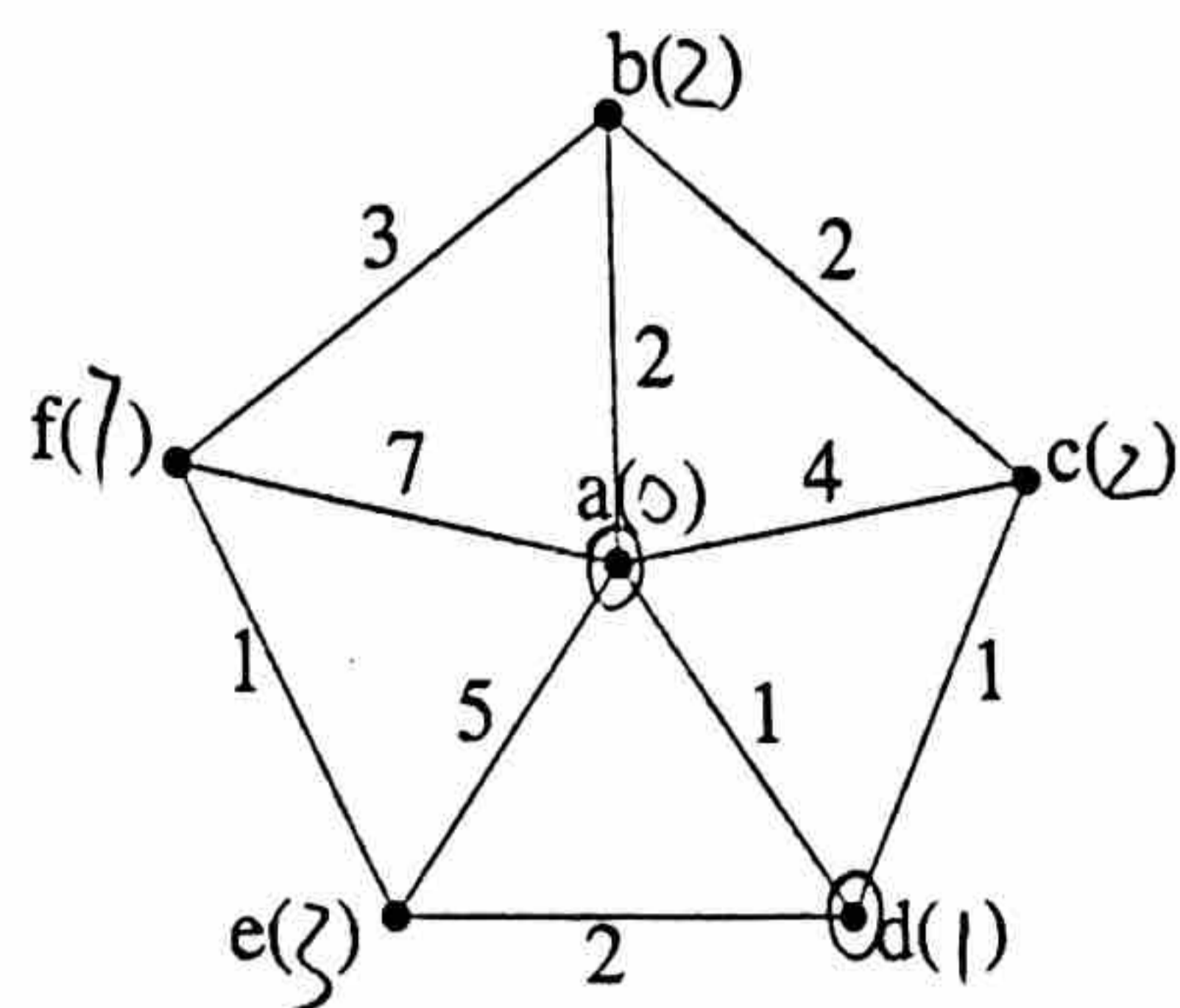
$$L(d) = \min\{0+1, \infty\} = 1$$

$$L(e) = \min\{0+5, \infty\} = 5$$

$$L(f) = \min\{0+7, \infty\} = 7$$

(go to the next page)

Second iteration:



Is f in T? Yes

current node: d

3/3

$$T = \{b, c, e, f\}$$

changes in labels: $L(c) = \min\{1+1, 4\} = 2$

$$L(e) = \min\{1+2, 5\} = 3$$