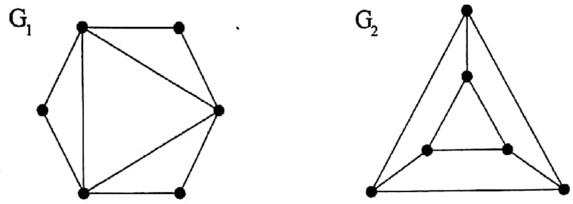
10

**1.** (10 points) **True or False:** Circle the right answers. You do NOT need to justify your answers.

Consider the following graphs:




The incidence matrices of $G_1$ and $G_2$ have the same number of rows. **T / F**

The incidence matrices of $G_1$ and $G_2$ have the same number of columns. **T / F**

$G_1$ and $G_2$ are isomorphic. **T / F**

$G_1$ and $G_2$ both have Hamiltonian cycles. **T / F**

$G_1$ admits an Euler cycle. **T / F**

$G_2$ admits an Euler cycle. **T / F**

$G_1$ admits an Euler path between different vertices. **T / F**

$G_2$ admits an Euler path between different vertices. **T / F**

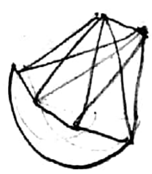
*Unrelated to the picture above:*

There exists a graph with six vertices, of degrees 4, 4, 4, 4, 4, 5. **T / F**

20 25

There exists a graph with six vertices, of degrees 4, 4, 4, 4, 5, 5. **T / F**

16 10 26



2. (6 points) Write down the answer to the following questions. *You do NOT need to justify your answers.*

If a graph has 100 vertices, all of degree 3, then how many edges does it have?

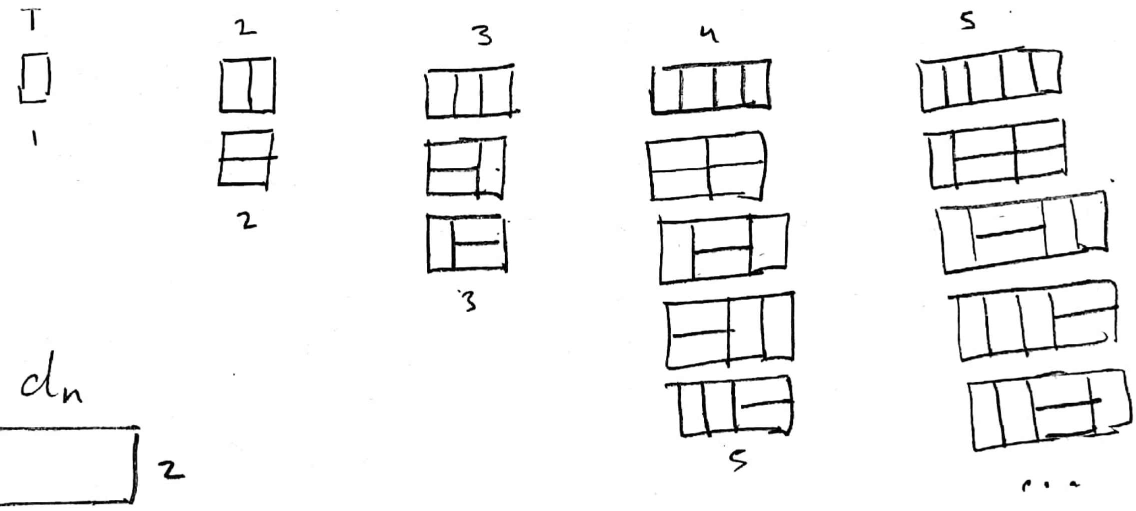
$$\sum_{k=0}^{n} \deg(k) = 2e$$

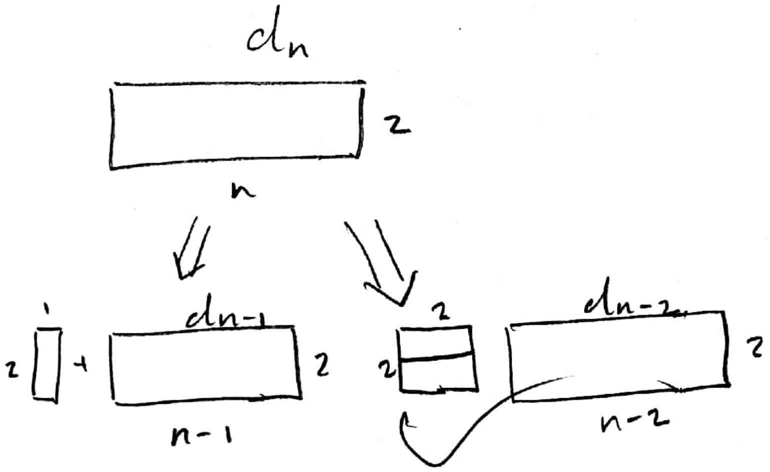
$$100 \cdot 3 = 300 = 2e$$

$$150 = e$$

$\boxed{150 \text{ edges}}$

Let $d_n$ be the number of ways one can fill a $2 \times n$ rectangle with $2 \times 1$ dominoes. Write down a recurrence relation for $d_n$.




$$\boxed{d_n = d_{n-1} + d_{n-2}}$$

each case can be split into adding a single domino to the previous case, or adding two horizontal dominos to the -2 case. Adding two vertical dominos are counted already since the n-1 case adds a single domino to the n-2 case, and we already added another domino to the n-1case

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**3.** (8 points) Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2}$$

with the initial conditions $a_0 = 2, a_1 = 2$. *Show all your work.*

$a_n = 2a_{n-1} + a_{n-2}$

$\Downarrow$

$t^2 = 2t + 1$

$t^2 - 2t - 1 = 0$

$t = \dfrac{-b \pm \sqrt{b^2 + 4ac}}{2a}$

$= \dfrac{2 \pm \sqrt{4+4}}{2}$

$= \dfrac{2 \pm 2\sqrt{2}}{2}$

$= 1 \pm \sqrt{2}$

solve homogenous linear equation

$a_n = b(1+\sqrt{2})^n + d(1+\sqrt{2})^n$

$a_0 = 2 = b + d$

$a_1 = 2 = b + b\sqrt{2} + d - d\sqrt{2}$

$b + d = b + d + b\sqrt{2} - d\sqrt{2}$

$0 = (b-d)\sqrt{2}$

$b = d$

$2 = b + d$

$2 = 2b$

$b = 1$

$d = 1$

solve for coefficients

$$\boxed{a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n}$$

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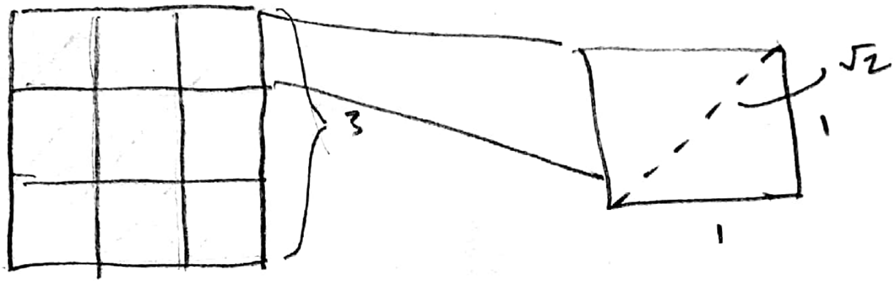
4. (8 points) Ten objects are placed on a square of side length 3 feet. Prove that there exist two objects at distance at most $\sqrt{2}$ feet from each other. *Show all your work.*

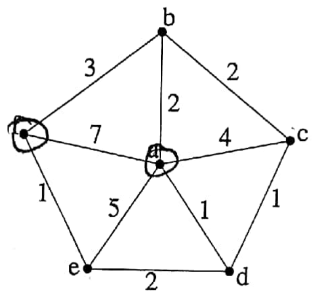



9 squares < 10 obj. → pigeonhole

A 3ft by 3ft square can be segmented into 9 1ft by 1ft squares. The diagonal of these squares, the longest line segment that fits in the square, is of length $\sqrt{2}$ ft. That means any items in the same square must be at most $\sqrt{2}$ feet apart. Since there are 9 squares, and 10 objects placed in the larger square, due to the pigeonhole principle, at least two objects must lie in the same square, and according to what we know earlier, they must be at most $\sqrt{2}$ ft apart.

8

**5.** (8 points) Consider the following weighted graph:

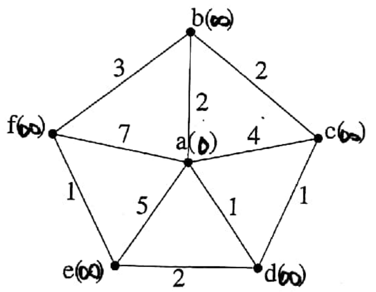



Suppose you want to use Dijkstra's algorithm to find the length of the shortest path from $a$ to $f$. Write down the initialization and the first two iterations of the algorithm. (You do NOT have to complete the whole algorithm.) At each step, write down if $f$ is in $T$, what is the current node, what is the new set of unvisited vertices $T$, then which nodes change labels and how.

Also, please circle the visited labels at each step (the ones not in $T$), and write down the label at each vertex in paranthesis.

**Initialization:**

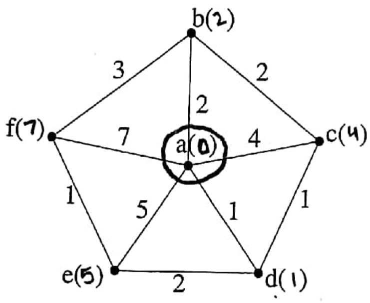
2/2




T= {a, b, c, d, e, f}

**First iteration:**

3/3




Is f in T? Yes

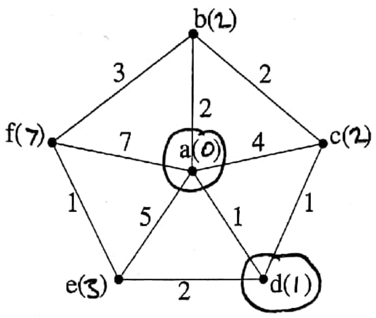
current node: a

T= {b, c, d, e, f}

changes in labels: b(∞ → 2) c(∞ → 4) d(∞ → 1) e(∞ → 5)
f(∞ → 7)

(go to the next page)

Second iteration:




313

Is f in T? Yes

current node: d

T= {b, c, e, f}

changes in labels: c (4 → 2) e (5 → 3)

*Do not write on this page.*

| | | |
|---|---|---|
| 1 | 10 | out of 10 points |
| 2 | 6 | out of 6 points |
| 3 | 8 | out of 8 points |
| 4 | 8 | out of 8 points |
| 5 | 8 | out of 8 points |
| Total | 40 | out of 40 points |