6

1. (8 points) **Circle the right answer. You do NOT need to justify your answers.**

(a) (1 point each) **True / False:** Consider the following relation on the set $\mathbb{Z}$ of integers:

$$xRy \iff x = y = 0.$$

Then:

R is reflexive — T / F

R is symmetric — T / F

R is antisymmetric — T / F

R is transitive — T / F

(b)(2 points each) **Multiple choice:** Consider the sequences a and b defined by

$$a_n = (-1)^n, \quad b_n = (-1)^{n+1}$$

a_n: $-1, 1, -1, 1$

b_n: $1, -1, 1, -1$

for $n \geq 1$. Then:

(A) a is a subsequence of b, but b is not a subsequence of a

(B) b is a subsequence of a, but a is not a subsequence of b

(C) a and b are subsequences of each other

(D) a is not a subsequence of b, and b is not a subsequence of a

In how many ways can we distribute 50 identical cookies to 7 distinct people, such that every person gets at least 2 cookies?

$(A)\ \frac{36!}{7! \cdot 29!} \quad (B)\ \frac{36!}{6! \cdot 30!} \quad (C)\ \frac{42!}{7! \cdot 35!} \quad (D)\ \frac{42!}{6! \cdot 36!}.$

2. (2 points each) Write down the answer to each question. **You do NOT need to justify your answers.** Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

However, the answer should be in closed form, not as a summation.

1

(a) Consider the relation R from $X = \{1, 2, 3\}$ to $Y = \{2, 3, 4\}$ given by

$$xRy \iff x < y.$$

Write down the matrix of R.

X

Y	1	2	3
2	1	0	0
3	1	1	0
4	1	1	1

$$= \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

0

(b) Consider a string of length n, made of n distinct letters. How many substrings does it have?

$= 2^n - (n-2)$ if $n \geq 2$

$= 2^n$ if $n \leq 1$

2

(c) Consider a sequence of length n, made of n distinct letters. How many subsequences does it have?

consecutive

$\boxed{= 2^n}$

a, b, c

ab ac bc

abc a b c

2^n

(d) Calculate the sum

$$\sum_{n=10}^{90} 2 \cdot (-3)^n.$$

$$\boxed{= \frac{2 \cdot (-3)^{n-8} - 1}{-4}}$$

0

(e) Consider the strings of length 10 made from the 26 letters of the English alphabet, such that they contain exactly three A's, exactly three B's, and no other letters are repeated. (An example of such a string is $BQWABAAXBM$.) How many such strings are there?

$$\boxed{= \binom{10}{3}\binom{7}{3}(24)(23)(22)(21)}$$

2

(f) Consider the strings of length 10 made from the 26 letters of the English alphabet, such that they contain at least one A and at least one B. (An example of such a string is $RTTADBLBBR$.) How many such strings are there?

$$= \binom{10}{1}\binom{9}{1} 26^8$$

0

$$= (10)(9)(26^8)$$

$$2 \cdot 26^8$$

$$26^{10} - 2(25)^{10} + (24)^{10}$$

A, B,

3. (a) (1 point) Write the definition of a one-to-one (injective) function.

Every value in the codomain must have at most one value from the domain mapped to it.

$\forall y \in Y$, there exists at most 1 $x \in X$ such that $f(x) = y$.

Next, answer the following questions, fully justifying your answers. (If an answer is *YES*, explain why. If an answer is *NO*, give a counterexample.)

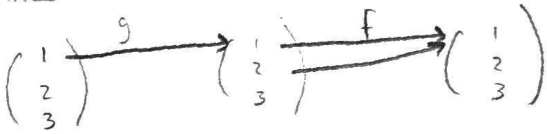
Suppose we have functions $g : X \to Y$ and $f : Y \to Z$, with composition $f \circ g : X \to Z$.

-1

(a) (3 points) If $f \circ g$ is one-to-one, does f have to be one-to-one?

No. Let X, Y, Z be the set $(1,2,3)$. Then, let $g : X \to Y = \{(1,1)\}$, and
$f : Y \to Z = \{(2,1), (1,1)\}$.

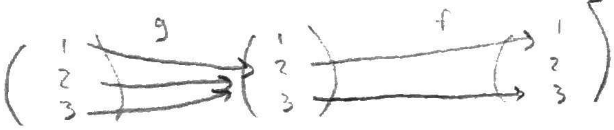
Clearly, $f \circ g$ is one-to-one, but f is not since 2 values in f's domain (2 and 1) point to the same value in the codomain.



-3

(b) (3 points) If f is one-to-one, does $f \circ g$ have to be one-to-one?

Yes. No matter what g is, the codomain of $f \circ g$ is determined by f (since it comes last), and if f is one-to-one, then no value in Z will be hit twice, so f and $f \circ g$ must be onto.



-1

(c) (3 points) If f and g are one-to-one, does $f \circ g$ have to be one-to-one?

Yes. For the same reason as above. If f is one-to-one, then the codomain of $f \circ g$ is determined by f, and if f is one-to-one, it means that no value in Z will be hit twice, so $f \circ g$ must also be one-to-one (since g must go through f to reach Z).

not quite correct

4. (10 points) Prove by induction on n that:

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1}n^2 = \frac{(-1)^{n+1}n(n+1)}{2}$$

for any integer $n \geq 1$.

Base case: $n=1$

$$(-1)^{1+1}(1)^2 \overset{?}{=} \frac{(-1)^{1+1}(1)(1+1)}{2}$$

$$1 = \frac{2}{2} = 1 \checkmark$$

Inductive step:

$$\underbrace{1^2 - 2^2 + 3^2 + (-1)^{n+1}n^2}_{\text{we assume this is equal to the original right side}} + (-1)^{n+2}(n+1)^2 \overset{?}{=} \frac{(-1)^{n+2}(n+1)(n+2)}{2}$$

$$\frac{(-1)^{n+1}n(n+1)}{2} + (-1)^{n+2}(n+1)^2 \overset{?}{=} \frac{(-1)^{(n+2)}(n+1)(n+2)}{2}$$

$$\frac{(-1)^{n+1}n\cancel{(n+1)} + 2(-1)^{n+2}(n+1)^{\cancel{2}}}{\cancel{2}} \overset{?}{=} \frac{(-1)^{n+2}\cancel{(n+1)}(n+2)}{\cancel{2}}$$

$$\frac{(-1)^{n+1}n + 2(-1)^{n+2}(n+1) \overset{?}{=} (-1)^{n+2}(n+2)}{(-1)^{n+2}}$$

(divide both sides by $(-1)^{n+2}$)

$$(-1)^{-1}n + 2(n+1) \overset{?}{=} n+2$$

$$-n + 2n + 2 \overset{?}{=} n + 2$$

$$n+2 = n+2 \checkmark$$

Hence, the statement is true for all $n \geq 1$ by induction.

Do not write on this page.

1	6	out of 8 points
2	5	out of 12 points
3	5	out of 10 points
4	10	out of 10 points
Total	26	out of 40 points