8

1. (8 points) **Circle the right answer. You do NOT need to justify your answers.**

(a) (1 point each) **True / False:** Consider the following relation on the set $\mathbb{Z}$ of integers:

$$xRy \iff x = y = 0.$$

Then:

R is reflexive   if $x=1$ $(x,x) \notin R$ T / F

R is symmetric   T / F

R is antisymmetric   T / F

R is transitive   T / F

(b)(2 points each) **Multiple choice:** Consider the sequences a and b defined by

$$a_n = (-1)^n, \quad b_n = (-1)^{n+1}$$

$a_n = \{-1, 1, -1, \dots\}$

$b_n = \{1, -1, 1, -1, \dots\}$

for $n \geq 1$. Then:

(A) a is a subsequence of b, but b is not a subsequence of a

(B) b is a subsequence of a, but a is not a subsequence of b

(C) a and b are subsequences of each other

(D) a is not a subsequence of b, and b is not a subsequence of a

In how many ways can we distribute 50 identical cookies to 7 distinct people, such that every person gets at least 2 cookies?

(A) $\frac{36!}{7! \cdot 29!}$. (B) $\frac{36!}{6! \cdot 30!}$ (C) $\frac{42!}{7! \cdot 35!}$ (D) $\frac{42!}{6! \cdot 36!}$.

give 2 cookies out first

50 − 14 cookies left

(36) $k = 36$ $n = 7$

$C(n+k-1, n-1) = C(42, 6) = \frac{42!}{6!\,36!}$

2. (2 points each) Write down the answer to each question. **You do NOT need to justify your answers.** Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

However, the answer should be in closed form, not as a summation.

2

(a) Consider the relation R from $X = \{1, 2, 3\}$ to $Y = \{2, 3, 4\}$ given by

$$xRy \iff x < y.$$

Write down the matrix of R.

$$Y$$

$$X \quad \begin{array}{c c} & \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

a b c d

$\{\lambda, a, b, c, d, ab, bc, cd, abc, bcd, abcd\}$ 11

0

(b) Consider a string of length n, made of n distinct letters. How many substrings does it have?

consecutive elements

n : 0, 1, 2, 3, 4

ss : 1, 2, 4, 7, 11

a b c

$\{\lambda, a, b, c, ab, bc, abc\}$ 7

$\boxed{n! + 1}$

a b

$\{\lambda, a, b, ab\}$ 4

2

1

2

(c) Consider a sequence of length n, made of n distinct letters. How many subsequences does it have?

doesn't have to be consecutive but has to be ordered

a b c

$= \{\lambda, a, b, c, ab, ac, bc, abc\}$

a b c d

$= \{\lambda, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, bcd, abcd\}$

$\boxed{2^n}$

$a \cdot \frac{r^{n+1} - 1}{r - 1}$

(d) Calculate the sum

$$\sum_{n=10}^{90} 2 \cdot (-3)^n.$$

$= 2 \sum_{n=10}^{90} (-3)^n$ let $n = j + 10$

$= 2 \sum_{j=0}^{80} (-3)^{j+10} = 2 \sum_{j=0}^{80} (-3)^{10} (-3)^j$

$= 2(-3)^{10} \cdot \frac{(-3)^{81} - 1}{-4} = \boxed{\frac{(-3)^{10} \cdot [(-3)^{81} - 1]}{-2}}$ 2

(e) Consider the strings of length 10 made from the 26 letters of the English alphabet, such that they contain exactly three A's, exactly three B's, and no other letters are repeated. (An example of such a string is $BQWABAAXBM$.) How many such strings are there?

$\binom{10}{3} \cdot \binom{7}{3} \cdot 24 \cdot 23 \cdot 22 \cdot 21$

$\binom{10}{3}$ → A, $\binom{7}{3}$ → B; $\frac{10!}{7!3!} \cdot \frac{7!}{4!3!}$

$= C(10,3) \cdot C(7,3) \cdot P(24,4)$

$= \boxed{\frac{10!}{3!3!4!} \cdot 24 \cdot 23 \cdot 22 \cdot 21}$ 2

(f) Consider the strings of length 10 made from the 26 letters of the English alphabet, such that they contain at least one A and at least one B. (An example of such a string is $RTTADBLBBR$.) How many such strings are there?

$\binom{10}{1} \cdot \binom{9}{1} \cdot 26^8$

$\binom{10}{1}$ → A, $\binom{9}{1}$ → B

$= \frac{10!}{9!1!} \cdot \frac{9!}{8!1!} \cdot 26^8$ 0

$= \boxed{\frac{10!}{8!} \cdot 26^8}$

Domain X, Codomain Y

3. (a) (1 point) Write the definition of a one-to-one (injective) function.

For every $y \in Y$, there are either zero or one $x \in X$ such that $(x, y) \in f$ (each $y \in Y$ gets mapped to by $f(x)$ at most one time).

Next, answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

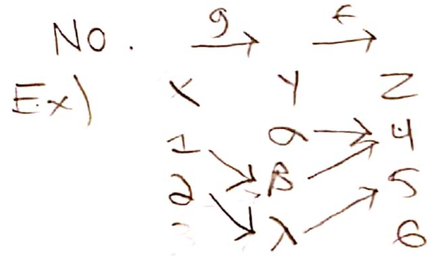
Suppose we have functions $g : X \to Y$ and $f : Y \to Z$, with composition $f \circ g : X \to Z$.

$X \xrightarrow{g} Y \xrightarrow{f} Z$

(a) (3 points) If $f \circ g$ is one-to-one, does f have to be one-to-one?

No.

Ex)



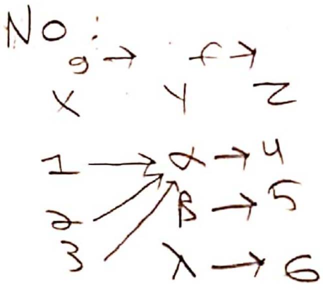
$g = \{(1, \beta), (2, \lambda)\}$

$f = \{(\alpha, 4), (\beta, 4), (\lambda, 5)\}$ → 4 gets mapped to twice

$f \circ g = \{(1, 4), (2, 5)\}$

(b) (3 points) If f is one-to-one, does $f \circ g$ have to be one-to-one?

No:



$g = \{(1, \alpha), (2, \alpha), (3, \alpha)\}$

$f = \{(\alpha, 4), (\beta, 5), (\lambda, 6)\}$ → each Z value gets mapped to once

$f \circ g = \{(1, 4), (2, 4), (3, 4)\}$ → each x value maps to the same Z value

(c) (3 points) If f and g are one-to-one, does $f \circ g$ have to be one-to-one?

-1

Yes. For g to be injective, each $y \in Y$ can have at most one $x \in X$ such that $g(x) = y$, so each $x \in X$ maps to a unique $y \in Y$. f then maps each of these unique $y \in Y$ to a unique $z \in Z$, and since x and z are arbitrary and $f \circ g$ needs to map $x \in X$ to a unique $z \in Z$, $f \circ g$ is also injective. close

4. (10 points) Prove by induction on n that:

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1}n^2 = \frac{(-1)^{n+1}n(n+1)}{2}$$

for any integer $n \geq 1$.

Base step: $n = 1$

$$1^2 \stackrel{?}{=} \frac{(-1)^{1+1}(1)(1+1)}{2}$$

$$1 \stackrel{?}{=} \frac{1 \cdot 2}{2} \qquad 1 = 1 \checkmark$$

Inductive step: $S(n) \Rightarrow S(n+1)$

Suppose $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1}n^2 = \frac{(-1)^{n+1}n(n+1)}{2}$

is true. Then,

$$\underbrace{1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1}n^2} + (-1)^{n+2}(n+1)^2 = \frac{(-1)^{n+2}(n+1)(n+2)}{2}$$

$$\frac{(-1)^{n+1}n(n+1)}{2} + \underbrace{(-1)^{n+2}}_{(-1)^{n+1}\cdot(-1)}(n+1)^2 \stackrel{?}{=} \boxed{\frac{(-1)^{n+2}(n+1)(n+2)}{2}}$$

$$= (-1)^{n+1}(n+1)\left[\frac{n}{2} - n - 1\right]$$

$$= (-1)^{n+1}(n+1)\left[\frac{n}{2} - \frac{2n}{2} - \frac{2}{2}\right]$$

$$= (-1)^{n+1}(n+1)\left[\frac{-n-2}{2}\right]$$

$$= \frac{(-1)^{n+1}(-1)(n+1)(n+2)}{2} \stackrel{\checkmark}{=} \boxed{\frac{(-1)^{n+2}(n+1)(n+2)}{2}}$$

By induction, the proof is complete