Math 61 : Discrete Structures
Midterm 1
Instructor: Ciprian Manolescu

You have 50 minutes.

No books, notes or calculators are allowed.
Do not use your own scratch paper.

1. (2 points each) **Multiple choice:** Circle the right answer. You do NOT need to justify your answers.

AB

A B

AB

× If α is a string of length two, what is the number of substrings of α?

(A) 2; (B) 3; (C) 4; (D) 5; (E) It depends on the string.

In how many ways can we distribute 12 distinct books to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12, 5)$; (D) $C(16, 4)$; (E) $C(16, 11)$.

$C(12, 5)$

✓ In how many ways can we distribute 12 (identical) copies of the same book to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12, 5)$; (D) $C(16, 4)$; (E) $C(16, 11)$.

$C(12+5-1, 4)$

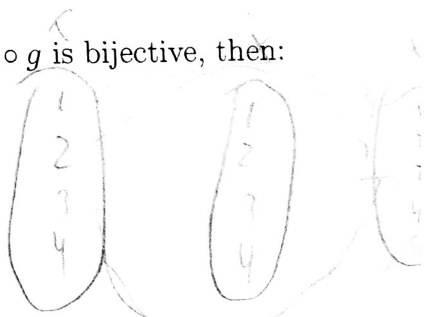
✓ Let $X = \{1, 2, 3, 4\}$. If $f, g : X \to X$ are two functions such that $f \circ g$ is bijective, then:

(A) f has to be bijective, but g does not have to be bijective;

(B) g has to be bijective, but f does not have to be bijective;

(C) Both f and g have to be bijective;

(D) Neither f nor g have to be bijective.



2. Write down the answer to each question. *You do NOT need to justify your answers.* Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

(a) Consider the set $X = \{1, 2, \ldots, 10\}$.

(2 points) How many of the relations on X are NOT reflexive?

Total − reflexive

$$\boxed{2^{100} - 10!^{10}}$$ ✗

(2 points) How many of the relations on X are symmetric?

$$\boxed{2^{\frac{10(11)}{2}}}$$ ✓

(2 points) How many of the relations on X are symmetric OR reflexive (i.e. either symmetric, or reflexive, or both)? Inc. Exc.

$$2^{\frac{10(11)}{2}} + \left(2^{100} - 10!^{10}\right) -$$ ✗

(2 points) How many of the relations on X are symmetric but NOT reflexive?

$$\boxed{2^{\frac{10(11)}{2}} - \left(2^{100} - 10!^{10}\right)}$$ ✗

(2 points) How many of the relations on X are both symmetric AND antisymmetric?

Total - non - antisym

$$2^{\frac{10(11)}{2}} + \left(2^{100} - ?\right)$$

+

(b) (2 points) How many distinct strings can be obtained from the string $AAABBCCCD$ by permuting (re-ordering) its letters?

AAABBCCCD

$$\boxed{\frac{9!}{3!\,2!\,3!}}$$

3 As

2 Bs

3 Cs

1 D

✓

$P(\mathbb{Z}) = \{\{1,2\}, \{2,3\}, \ldots$

3. Let $\mathbb{Z}$ be the set of all integers, and $\mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$ (consisting of all subsets of $\mathbb{Z}$). Consider the following relation R on $\mathcal{P}(\mathbb{Z})$: $\{1\}$ $\{(1,1)\}$

$$(A, B) \in R \iff A \cap B \neq \emptyset.$$

if and only if they do intersect

Answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

(a) (2 points) Is R reflexive?

2 $\boxed{\text{No.}}$ If we take the empty set, which is included in the power set of $\mathbb{Z}$, we find that $\emptyset \cap \emptyset$ is, in fact, equal to the empty set $(\emptyset)$, so R is not reflexive.

(b) (2 points) Is R symmetric?

2 $\boxed{\text{Yes.}}$ This relation holds if and only if $A \cap B \neq \emptyset$. The inverse would be $B \cap A \neq \emptyset$, which also holds in the same situations. $R = R^{-1}$, so R is symmetric.

(c) (3 points) Is R transitive?

3 $\boxed{\text{No}}$ Counterexample: Take the subsets $\{1,2\}$, $\{2,3\}$ and $\{3,4\}$. $\{1,2\} R \{2,3\}$ holds, and $\{2,3\} R \{3,4\}$ holds, but $\{1,2\} R \{3,4\}$ does not hold.

(d) (3 points) Prove that $(A, B) \in R \circ R$ whenever A and B are nonempty.

1 $R \circ R$ can be thought of in a transitive way. Let us again inspect $\{1,2\}$, $\{2,3\}$, and $\{3,4\}$. While in the original R, $\{1,2\} \not R \{3,4\}$, composing R with itself would make a new relation which is "transitive" compared to the last one. With these conditions, the only requirements that A and B need satisfy is for $A \neq \emptyset$ and $B \neq \emptyset$.

not a rigorous proof!

5·4·3·2
20
×6
120

4. (10 points) Prove by induction on n that:

$$3^n \leq (n+1)!$$

for any integer $n \geq 4$.

Base $n=4$

$3^4 \leq (4+1)! \rightarrow 81 \leq 120$ ✓

Induction Assume $3^k \leq (k+1)!$ for all $k \geq 4, k \in$

Then, $S(k+1) = S_k + a_{k+1}$

$3^{k+1} \leq (n+2)! \longrightarrow 3^k 3^1 \leq (k+2)!$

$3^k 3^1 \leq 3(k+1)!$ From here, we know that $(k+2)! = (k+2)(k+1)!$

So,

$3^k 3^1 \leq 3(k+1)! \leq (k+2)(k+1)!$

$(k+2)$ will always be a larger coefficient than 3, as k begins at $k=4$. This is how we can make the assumption that $3(k+1)! \leq (k+2)(k+1)!$

Finally, we can assume that $3^n \leq (n+1)!$ for any integer $n \geq 4$.

✓

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