Name: ______________________________

UCLA ID Num ______________________________

Section letter: 1C

Math 61 : Discrete Structures

Midterm 1

Instructor: Ciprian Manolescu

You have 50 minutes.

No notes, books or calculators are allowed.

Do not use your own scratch paper.

1. (2 points each) **Multiple choice:** Circle the right answer. You do NOT need to justify your answers.

If α is a string of length two, what is the number of substrings of α?

~~(A) 2;~~ (B) 3; (C) 4; (D) 5; (E) It depends on the string.

ab → λ, a, b, ab
1 2 3 4

✗

k n

In how many ways can we distribute 12 (identical) copies of the same book to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12, 5)$; (D) $C(16, 4)$; (E) $C(16, 11)$.

A..A | B..B | C..C | D..D | E..E

$\underbrace{* \cdots *}_{12 \text{ book}}$ ||||

choose 4 bars out of 16 position $C(16,4)$

✓

In how many ways can we distribute 12 distinct books to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12, 5)$; (D) $C(16, 4)$; (E) $C(16, 11)$.

$\frac{5}{1}\ \frac{5}{2}\ \frac{5}{3} \cdots \frac{5}{12} = 5^{12}$

book

✓

Let $X = \{1, 2, 3, \ldots\}$ be the set of all natural numbers. If $f, g : X \to X$ are two functions such that $f \circ g$ is bijective, then:

• onto
• one-to-one

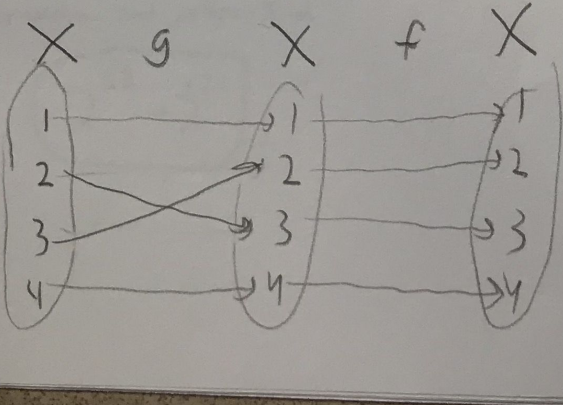
(A) f has to be bijective, but g does not have to be bijective;

(B) g has to be bijective, but f does not have to be bijective;

(C) Both f and g have to be bijective;

✗

(D) Neither f nor g have to be bijective.



49-7

2^{n^2-n}

# of spots - diagonal = n^2-n

reflexive = 2^{n^2-n}

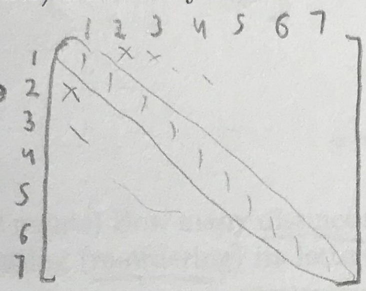
2. Write down the answer to each question. *You do NOT need to justify your answers.* Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

(a) Consider the set $X = \{1, 2, \ldots, 7\}$. $n=7$

(2 points) How many of the relations on X are NOT reflexive? =

n^2

reflexive →



• to be reflexive, all diagonals must be 1

• # reflexive relationships = $2^{n^2-n} = 2^{49-7} = 2^{42}$

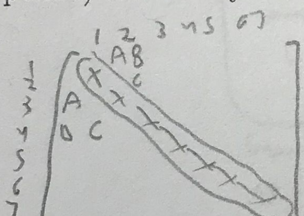
• all relationships = $2^{n^2} = 2^{49}$

$\boxed{2^{49}-2^{42}}$ ← answer

(2 points) How many of the relations on X are symmetric?

X = 0 or 1

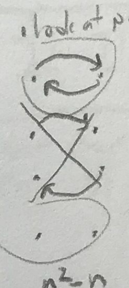
$\frac{49-7}{2} = \frac{42}{2} = 21$



$\frac{n^2-n}{2}$ $2^{21} \cdot 2^n = 2^{21} \cdot 2^7$

• look at pairs

2 options for each pair



$\frac{n^2-n}{2}$ pairs

$\underbrace{2^{\frac{n^2-n}{2}}}_{\text{pair}} \cdot \underbrace{2^n}_{\text{diagonal}} = 2^{21} \cdot 2^7$

$\boxed{2^{28}}$ ← answer

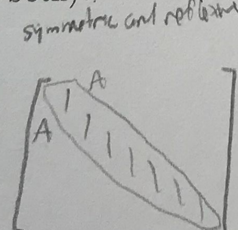
(2 points) How many of the relations on X are symmetric OR reflexive (i.e. either symmetric, or reflexive, or both)?

symmetric = 2^{28}

reflexive = 2^{42}

both = 2^{21}

symmetric and reflexive



$2^{\frac{n^2-n}{2}} = 2^{21}$

$\boxed{2^{28}+2^{42}-2^{21}}$

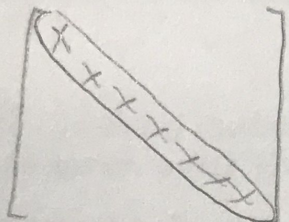
(2 points) How many of the relations on X are symmetric but NOT reflexive?

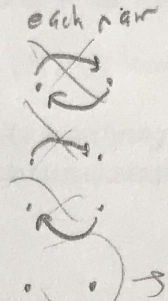
symmetric = 2^{28}

symmetric and reflexive = 2^{21}

$\boxed{2^{28}-2^{21}}$

(2 points) How many of the relations on X are both symmetric AND antisymmetric?





$\boxed{2^7}$

only freedom is diagonal

(b) (2 points) How many distinct strings can be obtained from the string $\overset{1\,2\,3\,4\,5\,6\,7\,8\,9}{AAAABCCDD}$ by permuting (re-ordering) its letters?

$$\boxed{\frac{9!}{\underbrace{4!}_{A's}\ \underbrace{2!}_{C's}\ \underbrace{2!}_{D's}}}$$

3. Let $\mathbb{Z}$ be the set of all integers, and $\mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$ (consisting of all subsets of $\mathbb{Z}$). Consider the following relation R on $\mathcal{P}(\mathbb{Z})$: intersection

$$(A, B) \in R \iff A \cap B = \emptyset.$$ A and B have no elements in common

Answer the following questions, fully justifying your answers. (If an answer is *YES*, explain why. If an answer is *NO*, give a counterexample.)

(a) (2 points) Is R reflexive?

No.

Counterexample: $A = \{1\}$

$(A, A) \notin R$ since $\{1\} \cap \{1\} = \{1\} \neq \emptyset$

(b) (2 points) Is R symmetric?

Yes.

Intersection is commutative. $A \cap B$ is the same thing as $B \cap A$. Both will contain the elements that they both contain. Therefore, if $(A, B) \in R$, $(B, A) \in R$ as well.

(c) (3 points) Is R transitive?

No.

Counterexample: $A = \{1\}$, $B = \{2\}$, $C = \{1, 3\}$

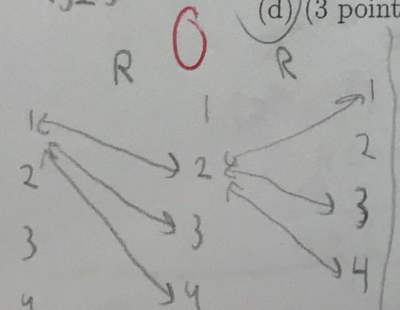
- $(A, B) \in R$ since $\{1\} \cap \{2\} = \emptyset$
- $(B, C) \in R$ since $\{2\} \cap \{1,3\} \neq \emptyset$

However, $(A, C) \notin R$ since $\{1\} \cap \{1,3\} = \{1\} \neq \emptyset$

(1,2)

0

(d) (3 points) Prove that $(A, B) \in R \circ R$ for all A and B.



Say $(A, B) \in R$. That means $A \cap B = \emptyset$ (A and B have no elements in common). So now, $R \circ R$ will test if $\cancel{(A \cap B)} \cap (A \cap B)$ is $\in R$. Since $A \cap B \neq \emptyset$, and $\emptyset \cap X$, where X is any possible set, $= \emptyset$, then $R \circ R$ will always $= \emptyset$, which proves $(A, B) \in R \circ R$ for all A and B.

$\emptyset \cap \dots = \emptyset$

10

4. (10 points) Prove by induction on n that:

$$2^{2n} \leq (n+2)!,$$

for any integer $n \geq 1$. Use Mathematical Induction

Base Case

$n = 1$

$2^{2(1)} \leq (1+2)!$

$2^2 \leq 3!$

$4 \leq 6 \rightarrow$ true, Base Case is true.

Induction Step

$S(n)$: Assume $2^{2n} \leq (n+2)!$ for $n \geq 1$

$S(n+1)$: ~~Then~~ $2^{2(n+1)} \leq (n+1+2)!$

we want to prove $\longrightarrow$ $2^{2n+2} \leq (n+3)!$ Note: $(n+3)! = (n+3)(n+2)!$

$2^{2n+2} \leq (n+3)(n+2)!$

• If we multiply both sides of the assumption by $(n+3)$, we get

$(n+3)2^{2n} \leq (n+3)!$

Therefore, it suffices to show:

$2^{2n+2} \leq (n+3)2^{2n}$

$(\cancel{2^{2n}})(2^2) \leq (n+3)\cancel{2^{2n}}$

$4 \leq n+3$

$-3 \quad -3$

$n \geq 1 \rightarrow$ true since $n \geq 1$ for this inequality

Therefore, since $2^{2n+2} \leq (n+3)2^{2n} \leq (n+3)!$, that means $2^{2n+2} \leq (n+3)!$, so $S(n+1)$ is true for $n \geq 1$, which proves $S(n)$ is true for any integer $n \geq 1$.

any integer

Do not write on this page.

	1	4	out of 8 points
16	2	12	out of 12 points
	3	7	out of 10 points
17	4	10	out of 10 points
	Total	33	out of 40 points