1. (2 points each) **Multiple choice:** Circle the right answer. You do NOT need to justify your answers.

If α is a string of length two, what is the number of substrings of α?

(A) 2; (B) 3; (C) 4; (D) 5; (E) It depends on the string.

✓

ab

aa

no repetition distinct

In how many ways can we distribute 12 distinct books to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12,5)$; (D) $C(16,4)$; (E) $C(16,11)$.

✓

Step 1

Give book 1 to 1 of 5 people: 5 ways

Step 2

Give book 2 to 1 of 5 people: 5 ways

In how many ways can we distribute 12 (identical) copies of the same book to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12,5)$; (D) $C(16,4)$; (E) $C(16,11)$.

✓

12+5-1,

16

* * * * * * * * * * * *

c(16, 4)

Let $X = \{1, 2, 3, 4\}$. If $f, g : X \to X$ are two functions such that $f \circ g$ is bijective, then:

(A) f has to be bijective, but g does not have to be bijective;

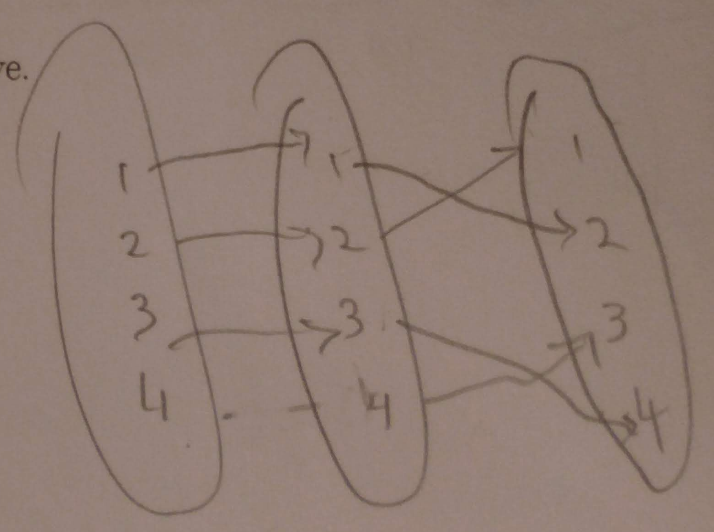
(B) g has to be bijective, but f does not have to be bijective; ✓

(C) Both f and g have to be bijective;

(D) Neither f nor g have to be bijective.

$f: X \to X$

$g: X \to X$



$\binom{\;\;}{\;\;}$ 2^{a} relations

2. Write down the answer to each question. *You do NOT need to justify your answers.* Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

(a) Consider the set $X = \{1, 2, \ldots, 10\}$.

(2 points) How many of the relations on X are NOT reflexive?

1 2 . . . 10

1
2
10

2^{100} total relations

2^{10^2-10} are reflexive

$\boxed{2^{100} - 2^{90}}$ ✓

(2 points) How many of the relations on X are symmetric?

$\left(2^{\frac{n^2-n}{2}}\right) \cdot 2^{n} = 2^{\frac{n^2+n}{2}} = 2^{110/2} = \boxed{2^{55}}$ ✓

(2 points) How many of the relations on X are symmetric OR reflexive (i.e. either symmetric, or reflexive, or both)?

sym. & refl.

$2^{\frac{n^2-n}{2}} = 2$

$\boxed{\underbrace{2^{55}}_{\text{sym.}} + \underbrace{2^{100} - 2^{90}}_{\text{refl.}} - \underbrace{2^{45}}_{\text{sym. \& refl.}}}$ ×

(2 points) How many of the relations on X are symmetric but NOT reflexive?

$\boxed{2^{55} - 2^{45}}$ ✓

S R

(2 points) How many of the relations on X are both symmetric AND antisymmetric?

$2^n \rightarrow \boxed{2^{10}}$

either 1's or 0's on diagonal

(b) (2 points) How many distinct strings can be obtained from the string $AAABBCCCD$ by permuting (re-ordering) its letters?

$$\frac{9!}{3!\,2!\,3!}$$

3. Let $\mathbb{Z}$ be the set of all integers, and $\mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$ (consisting of all subsets of $\mathbb{Z}$). Consider the following relation R on $\mathcal{P}(\mathbb{Z})$:

$$(A, B) \in R \iff A \cap B \neq \emptyset.$$

Answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

(a) (2 points) Is R reflexive?

2

no, let $A = \emptyset$. Then $A \cap A = \emptyset$. Hence, this does not satisfy the relation

(b) (2 points) Is R symmetric?

2

yes, the ~~union~~ of 2 sets is commutitive. $A \cap B$ is the same as $B \cap A$. Therefore, if $(A, B) \in R$, then $(B, A) \in R$ as well

(c) (3 points) Is R transitive?

3

No. Let $A = \{3\}$, $B = \{3, 5\}$, $C = \{5, 7\}$. Then ARB (since $A \cap B = \{3\}$), BRC (since $B \cap C = \{5\}$) but $A \not R C$ (since $A \cap C = \emptyset$)

(d) (3 points) Prove that $(A, B) \in R \circ R$ whenever A and B are nonempty.

$R \circ R = A^2$, where A is the matrix of the relation

Then, if A and B are non empty, they will be in A^2

what is the matrix??

4. (10 points) Prove by induction on n that:

$$3^n \leq (n+1)!$$

for any integer $n \geq 4$.

base case ($n=4$)

LHS

$3^4 = 81$

RHS

$(4+1)! = 5! = 120$

$81 \leq 120$ ✓

Inductive step

Suppose $3^n \leq (n+1)!$. We want to show that

$3^{n+1} \leq ((n+1)+1)!$ . . .

$((n+1)+1)! = (n+2)! = (n+2)(n+1)!$

using the inductive hypothesis,

$3^n (n+2) \leq (n+1)! \cdot (n+2)$.

Therefore, it suffices to show $3^{n+1} \leq 3^n (n+2)$

$3^{n+1} = 3 \cdot 3^n$, which is $< 3^n (n+2)$ since

$n+2 > 3$ for $n \geq 4$.

$\therefore$ we have showed that $3^n \leq (n+1)!$ for all $n \geq 4$, $n \in \mathbb{Z}$

✓