1. (2 points each) **Multiple choice:** Circle the right answer. You do NOT need to justify your answers.

ab → λ a b ab

aa → λ a aa

If α is a string of length two, what is the number of substrings of α?

(A) 2; (B) 3; (C) 4; (D) 5; (E) It depends on the string.

✓

In how many ways can we distribute 12 distinct books to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12, 5)$; (D) $C(16, 4)$; (E) $C(16, 11)$.

✓

In how many ways can we distribute 12 (identical) copies of the same book to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12, 5)$; (D) $C(16, 4)$; (E) $C(16, 11)$. ✓

12+4, 4

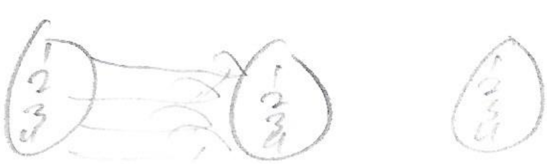
Let $X = \{1, 2, 3, 4\}$. If $f, g : X \to X$ are two functions such that $f \circ g$ is bijective, then:

(A) f has to be bijective, but g does not have to be bijective;

(B) g has to be bijective, but f does not have to be bijective;

(C) Both f and g have to be bijective; ✓

(D) Neither f nor g have to be bijective.



2. Write down the answer to each question. *You do NOT need to justify your answers.* Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

(a) Consider the set $X = \{1, 2, \ldots, 10\}$.

$n = 10$

(2 points) How many of the relations on X are NOT reflexive?

reflexive $= 2^{n^2-n}$

$2^{n^2} - 2^{n^2-n}$

$2^{100} - 2^{90}$ ✓

(2 points) How many of the relations on X are symmetric?

$2^{n(n+1)/2} = 2^{10(11)/2} = 2^{55}$ ✓

(2 points) How many of the relations on X are symmetric OR reflexive (i.e. either symmetric, or reflexive, or both)?

$2^{n^2-n} + 2^{n(n+1)/2} - 2^{n(n-1)/2}$

$2^{90} + 2^{55} - 2^{45}$ ✓

(2 points) How many of the relations on X are symmetric but NOT reflexive?

Sym: $2^{n(n+1)/2}$ Sym and ref: $2^{n(n-1)/2}$

Sym + not ref $= 2^{n(n+1)/2} - 2^{n(n-1)/2}$

$2^{55} - 2^{45}$ ✓

(2 points) How many of the relations on X are both symmetric AND antisymmetric?

$2^n = 2^{10}$

✓

1 2 3 4 5 6 7 8 9

(b) (2 points) How many distinct strings can be obtained from the string $AAABBCCCD$ by permuting (re-ordering) its letters?

$$\frac{9!}{3!2!3!}$$

✓

3. Let $\mathbb{Z}$ be the set of all integers, and $\mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$ (consisting of all subsets of $\mathbb{Z}$). Consider the following relation R on $\mathcal{P}(\mathbb{Z})$:

$$(A, B) \in R \iff A \cap B \neq \emptyset.$$

Answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

empty set

(a) (2 points) Is R reflexive?

no, if $A = \{\}$, then $A \cap A = \emptyset$ and $(A, A) \notin R$ for all $A \in P(\mathbb{Z})$

(b) (2 points) Is R symmetric?

yes, since if $A \cap B \neq \emptyset$, then $B \cap A \neq \emptyset$, so if $(A, B) \in R$, $(B, A) \in R$

(c) (3 points) Is R transitive?

no, consider $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$

$A \cap B \neq 0$, $B \cap C \neq 0$, but $A \cap C = \emptyset$, so $(A, B) \in R$, $(B, C) \in R$, but $(A, C) \notin R$

(d) (3 points) Prove that $(A, B) \in R \circ R$ whenever A and B are nonempty.

Since A is nonempty and $P(\mathbb{Z})$ consists of all subsets of $\mathbb{Z}$, there is a subset, denoted C, which contains all integers $\in \mathbb{Z}$. Therefore $A \cap C \neq \emptyset$ and $(A, C) \in R$. B is also nonempty, so $C \cap B \neq \emptyset$ and $(C, B) \in R$. Therefore composing $R \circ R$ gives

$A \xrightarrow{R} C \xrightarrow{R} B$, $A \xrightarrow{R \circ R} B$

and $(A, B) \in R \circ R$ for any nonempty A and B.

4. (10 points) Prove by induction on n that:

$$3^n \leq (n+1)!$$

for any integer $n \geq 4$.

base case: $n=4 \to 3^4 \leq (5)!$

$81 \leq 5\cdot4\cdot3\cdot2\cdot1$

$81 \leq 20\cdot6$

$81 \leq 120$ ✓

assume $3^n \leq (n+1)!$ is true, then

$3^{n+1} \leq 3\cdot(n+1)! \leq (n+2)!$

It suffices to show that

$3(n+1)! \leq (n+2)!$ for $n \geq 4$

$3\cdot(n+1)! \leq (n+2)(n+1)!$

$3 \leq n+2$, when $n \geq 4$ this is true

since $1 \leq n$ ✓

$\therefore 3^n \leq (n+1)!$ for $n \geq 4$

✓ 10