71.

1. (2 points each) **Multiple choice:** Circle the right answer. You do NOT need to justify your answers.

If α is a string of length two, what is the number of substrings of α?

(A) 2; (B) 3; (C) 4; (D) 5; (E) It depends on the string.

X

In how many ways can we distribute 12 (identical) copies of the same book to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12,5)$; (D) $C(16,4)$; (E) $C(16,11)$. ✓

In how many ways can we distribute 12 distinct books to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12,5)$; (D) $C(16,4)$; (E) $C(16,11)$. ✓

Let $X = \{1, 2, 3, \dots\}$ be the set of all natural numbers. If $f, g : X \to X$ are two functions such that $f \circ g$ is bijective, then:

(A) f has to be bijective, but g does not have to be bijective;

(B) g has to be bijective, but f does not have to be bijective; X

(C) Both f and g have to be bijective;

(D) Neither f nor g have to be bijective.

2. Write down the answer to each question. *You do NOT need to justify your answers.* Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

(a) Consider the set $X = \{1, 2, \ldots, 7\}$.

(2 points) How many of the relations on X are NOT reflexive?

$2^{49} - 2^{42}$ ✓

(2 points) How many of the relations on X are symmetric?

$2^{\frac{7(7+1)}{2}} = 2^{28}$ ✓

(2 points) How many of the relations on X are symmetric OR reflexive (i.e. either symmetric, or reflexive, or both)?

$2^{28} + 2^{42} - \left(2^{\frac{7(7-1)}{2}}\right)$

$= 2^{28} + 2^{42} - 2^{21}$ ✓

(2 points) How many of the relations on X are symmetric but NOT reflexive?

$2^{28} - 2^{\frac{7(7-1)}{2}} = 2^{28} - 2^{21}$ ✓

(2 points) How many of the relations on X are both symmetric AND antisymmetric?

2^7

(b) (2 points) How many distinct strings can be obtained from the string $AAAABCCDD$ by permuting (re-ordering) its letters?

$$\frac{9!}{4!2!2!}$$

3. Let $\mathbb{Z}$ be the set of all integers, and $\mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$ (consisting of all subsets of $\mathbb{Z}$). Consider the following relation R on $\mathcal{P}(\mathbb{Z})$:

$$(A, B) \in R \iff A \cap B = \emptyset.$$

Answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

(a) (2 points) Is R reflexive?

No. Let $A = \{1, 2\}$

$A \cap A = A = \{1, 2\} \neq \emptyset$

(b) (2 points) Is R symmetric?

Yes. If $A \cap B = \emptyset$

Then by commutativity of $\cap$, $B \cap A = \emptyset$

$\therefore (B, A) \in R$.

(c) (3 points) Is R transitive?

No. Let $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{1, 5\}$

$A \cap B = \emptyset$, $B \cap C = \emptyset$

but $A \cap C = \{1\} \neq \emptyset$ $\therefore (A, C) \notin R$ when $(A, B) \in R$ and $(B, C) \in R$

(d) (3 points) Prove that $(A, B) \in R \circ R$ for all A and B.

By def. of $R \circ R$, for each $A, B \in P(\mathbb{Z})$

s.t. $(A, B) \in R \circ R$, $\exists C \in P(\mathbb{Z})$ s.t. $(A, C) \in R$ and $(C, B) \in R$

$\iff A \cap C = \emptyset$ and $C \cap B = \emptyset$

We simply let $C = \emptyset$ and by property of $\emptyset$

$A \cap \emptyset = \emptyset$ and $\emptyset \cap B = \emptyset$ for all $A, B \in P(\mathbb{Z})$

$\therefore (A, B) \in R \circ R$ for all A and B.

4. (10 points) Prove by induction on n that:

$$2^{2n} \leq (n+2)!,$$

for any integer $n \geq 1$.

Let P_1 : $2^2 = 4 \leq 6 = 3! = (1+2)!$

·Assume P_n is true

We check P_{n+1} is also true :

P_{n+1} : $2^{2(n+1)} = 2^{2n} \cdot 2^2 \leq (n+2)! \cdot 2^2$

$\therefore$ we want to show $(n+2)! \cdot 2^2 = 4(n+2)! \leq (n+3)!$ for $n \geq 1$

$\iff 4 \leq n+3$

$\iff 1 \leq n$

Since $n \geq 1$, the inequality above is satisfied.

$\therefore$ $2^{2(n+1)} \leq (n+2)! \cdot 2^2 \leq (n+3)! = ((n+1)+2)!$ ✓

10

$\therefore$ $2^{2n} \leq (n+2)!$ for all $n \geq 1$