Name: ________________________________

UCLA I________________________________

Section I

Math 61 : Discrete Structures
Midterm 1
Instructor: Ciprian Manolescu

You have 50 minutes.

No notes, books or calculators are allowed.
Do not use your own scratch paper.

1. (2 points each) **Multiple choice:** Circle the right answer. You do NOT need to justify your answers.

If α is a string of length two, what is the number of substrings of α?

(A) 2; (B) 3; (C) 4; (D) 5; (E) It depends on the string.

In how many ways can we distribute 12 (identical) copies of the same book to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12,5)$; (D) $C(16,4)$; (E) $C(16,11)$.

In how many ways can we distribute 12 distinct books to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12,5)$; (D) $C(16,4)$; (E) $C(16,11)$.

Let $X = \{1,2,3,\dots\}$ be the set of all natural numbers. If $f, g : X \to X$ are two functions such that $f \circ g$ is bijective, then:

(A) f has to be bijective, but g does not have to be bijective;

(B) g has to be bijective, but f does not have to be bijective;

(C) Both f and g have to be bijective;

(D) Neither f nor g have to be bijective.

2. Write down the answer to each question. *You do NOT need to justify your answers.* Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

(a) Consider the set $X = \{1, 2, \ldots, 7\}$.

(2 points) How many of the relations on X are NOT reflexive?

total 2^{49} relations, and 2^{42} relations that are reflexive

$$\boxed{2^{49} - 2^{42}} = 2^{42}(2^7-1) = 2^{42}(127)$$

✓

(2 points) How many of the relations on X are symmetric?

$2^{\frac{7(7-1)}{2}}$ choices to make symmetric, 2^7 choices for xRx (diagonal)

$$2^7 \cdot 2^{21} = \boxed{2^{28}}$$

✓

(2 points) How many of the relations on X are symmetric OR reflexive (i.e. either symmetric, or reflexive, or both)?.

$|\text{Symmetric} \cup \text{reflexive}| = |S| + |R| - |S \cap R|$

$$2^{28} + (2^{49} - 2^{42}) - 2^{21}$$

$$2^{49} + 2^{28} - 2^{42} - 2^{21}$$

$$2^{28}(2^{21}+1) - 2^{21}(2^{21}+1)$$

✗

$$\boxed{(2^{21}+1)(2^{28}-2^{21})} = 2^{21}(2^{21}+1)(2^7-1)$$

(2 points) How many of the relations on X are symmetric but NOT reflexive?

$|S| - |S \cap R|$

$$\boxed{2^{28} - 2^{21}} = 2^{21}(2^7-1) = 127 \cdot 2^{21}$$

✓

(2 points) How many of the relations on X are both symmetric AND antisymmetric?

All but diagonal must be 0, choose from diagonal entries

$\boxed{2^7}$

(b) (2 points) How many distinct strings can be obtained from the string $AAAABCCDD$ by permuting (re-ordering) its letters?

$\boxed{\frac{9!}{4!\,2!\,2!}}$

Scanned by CamScanner

3. Let $\mathbb{Z}$ be the set of all integers, and $\mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$ (consisting of all subsets of $\mathbb{Z}$). Consider the following relation R on $\mathcal{P}(\mathbb{Z})$:

$$(A, B) \in R \iff A \cap B = \emptyset.$$

Answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

(a) (2 points) Is R reflexive?

$\boxed{\text{NO.}}$ Let $A = \{1, 2, 3\}$

$A \cap A = A = \{1, 2, 3\} \neq \phi$, so $(A, A) \notin R$

(b) (2 points) Is R symmetric?

$\boxed{\text{Yes,}}$ if $(A, B) \in R \iff A \cap B = \emptyset \iff B \cap A = \emptyset \iff (B, A) \in R$

since if $\forall a \in A$, $a \notin B$ and $\forall b \in B$, $b \notin A$, then similarly $\forall b \in B$, $b \notin A$ and $\forall a \in A$, $a \notin B$

(c) (3 points) Is R transitive?

$\boxed{\text{NO.}}$ $A = \{1, 2\}$ $B = \{3, 4\}$ $C = \{1, 5\}$

$(A, B) \in R$ since $A \cap B = \emptyset$

$(B, C) \in R$ since $B \cap C = \emptyset$

$(A, C) \notin R$ since $A \cap C = \{1\} \neq \emptyset$

(d) (3 points) Prove that $(A, B) \in R \circ R$ for all A and B.

$(A, B) \in R \iff A \cap B = \emptyset$

$(A, B) \in R \circ R \iff \exists\, C$ s.t. $(A, C) \in R$ and $(C, B) \in R$

$A \cap C = \emptyset$ $\quad$ $C \cap B = \emptyset$

this is true for all A and B because we can just choose C to be the empty set $\emptyset$, then $A \cap C = \emptyset$ and $C \cap B = \emptyset$ are always satisfied

10

4. (10 points) Prove by induction on n that:

$$2^{2n} \leq (n+2)!,$$

for any integer $n \geq 1$.

Base case $n=1$ $\qquad 2^{2(1)} \leq (1+2)!$

$4 \leq 3! = 6$ true ✓

Inductive Step $\quad$ assume $2^{2n} \leq (n+2)!$

want $2^{2(n+1)} \leq ((n+1)+2)!$

$\Downarrow$

$2^{2n+2} \leq (n+3)!$

$2^2 \cdot 2^{2n} \leq (n+3)(n+2)!$

by our inductive hypothesis, it suffices to Show that $2^2 \leq (n+3)$

let $n \geq 1$, $\quad 4 \leq (1+3) = 4$

$4 \leq (n+3) \quad \forall n \geq 1$

therefore the statement on $n+1$ is true which implies $2^{2n} \leq (n+2)!$ on $n \geq 1$

Do not write on this page.

1	4	out of 8 points
2	10	out of 12 points
3	10	out of 10 points
4	10	out of 10 points
Total	34	out of 40 points