22W-MATH-61-LEC-1 / 22W-MATH-61-LEC-2 Exam 1--Written Portion

KEATON HEISTERMAN

TOTAL POINTS

66 / 66

QUESTION 1

1 Problem 1 0 / 0

✓ **- 0 pts** Correct

QUESTION 2

Problem 2 20 pts

2.1 2(a) 10 / 10

✓ **- 0 pts** Correct

- 5 pts Miscalculated floor function in counterexample.

- 5 pts Used non-integer inputs for $$f$$.

- 8 pts Attempted to prove that $$f$$ is one-to-one by performing invalid operations on the floor function.

- 10 pts Blank

2.2 2(b) 10 / 10

✓ **- 0 pts** Correct

- 0 pts See comments.

- 1 pts See comments

- 2 pts See comments

- 3 pts See comments

- 4 pts See comments

- 5 pts See comments

QUESTION 3

Problem 3 20 pts

3.1 3(a) 10 / 10

✓ **- 0 pts** Correct

- 0.5 pts Correct combinatorics; arithmetic error

- 1.5 pts Picked the pairs of cats in an order, and didn't correct for overcounting

- 1.5 pts Used C(4,2) to pick a single kitten

- 1.5 pts Within each pair, picked the kittens in order

- 10 pts Completely incorrect / nothing written

3.2 3(b) 10 / 10

✓ **- 0 pts Correct justification of the correct answer.**

- 1 pts Correct setup, but incorrect answer due to arithmetic error.

- 2 pts Incorrect setup, but only failed to distinguish the two orange kittens.

- 4 pts Incorrect approach or justification, but shows understanding of basic counting principles.

- 6 pts Only (correctly) justified the $$5!$$ possible naive arrangements and/or the $$3!$$ possible arrangements of the non-orange kittens.

- 8 pts Had some correct explanation, but did not show understanding of basic counting principles.

- 10 pts Failed to correctly justify any important part of the problem.

QUESTION 4

4 Problem 4 10 / 10

✓ **- 0 pts** Correct

- 2 pts Neglected to say that $$x\neq y$$

- 4 pts Counterexample is incorrect or missing

- 2 pts In the counterexample, $$R$$ is not symmetric

- 1 pts In the counterexample, $$R \cap S$$ is written incorrectly

- 6 pts The proof that $$R \cap S$$ is antisymmetric is missing

- 5 pts The proof of the first part is incorrect but the definition of antisymmetric relations is stated

- 1 pts Small inconsistency in explainaitions

- 2 pts A specific counter example is not provided, but it is explained how to obtain it

- **1 pts** In the proof of the first part, we start with $$(x,y)\in R$$ instead of $$(x,y) \in R\cap S$$

- **5 pts** The proof of the first part relies on $$R$$ being symmetric

- **1 pts** The conclusion $$x=y$$ for $$(x,y),(y,x) \in R\cap S$$ is missing

- **10 pts** Incorrect

QUESTION 5

5 Problem 5 **6 / 6**

✓ **- 0 pts** **Correct**

- **2 pts** Example is not reflexive

- **2 pts** Example is not transitive

- **2 pts** Example is not symmetric

- **1 pts** Minor error(s) or step missing (see comments)

- **1 pts** Did not state how many equivalence classes there are

- **2 pts** Equivalence relation is not containing (1,3) or (2,3)

- **1 pts** Did not say what the partition is, but did write the equivalence classes correctly somewhere in the work

- **2 pts** Incorrect or missing partition, and the correct equivalence classes are not written anywhere

- **0.5 pts** Notational error (see comments)

- **2 pts** Only gave the equivalence classes, but did not give the equivalence relation

QUESTION 6

6 Problem 6 **10 / 10**

✓ **- 0 pts** **Correct**

- **1 pts** Missing minor details/small mistakes

- **2 pts** Minor issues with solution

- **3 pts** Issues with solution

- **5 pts** Major issues with solution

- **7 pts** Some good ideas, but not a solution

- **8 pts** Some understanding of key concepts, but not a solution

Exam 1

name Keaton Heisterman Student ID #: 905 537 242

Section number Discussion 1 B

1. I, Keaton Heisterman, have read and have not used any non-permitted materials or recieved any help on this exam.

1 Problem 1 **0 / 0**

✓ **- 0 pts** Correct

Question 2 a)

The claim that f is oneto-one is false.

Proof:

A function f from X to Y is one-to-one if for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$. Thus to prove f is not one-to-one, we must show for some $x_1, x_2 \in \mathbb{Z}$ that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Observe: if $x_1 = 1$ and $x_2 = 2$, $x_1, x_2 \in \mathbb{Z}$. Now, see that:

$$f(x_1) = f(1) = \left\lfloor \frac{1+1}{7} \right\rfloor = \left\lfloor \frac{2}{7} \right\rfloor = 0 \text{, and,}$$

$$f(x_2) = f(2) = \left\lfloor \frac{2+1}{7} \right\rfloor = \left\lfloor \frac{3}{7} \right\rfloor = 0$$

Thus we see that $f(x_1) = f(x_2)$, but, $x_1 \neq x_2$ for some $x_1, x_2 \in \mathbb{Z}$

Therefore we haved proved that $f: \mathbb{Z} \to \mathbb{Z}$ is not one-to-one because for some $x_1, x_2 \in \mathbb{Z}$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$ □

2.1 2(a) **10 / 10**

✓ **- 0 pts** **Correct**

- 5 pts Miscalculated floor function in counterexample.

- 5 pts Used non-integer inputs for $$f$$.

- 8 pts Attempted to prove that $$f$$ is one-to-one by performing invalid operations on the floor function.

- 10 pts Blank

Question 2 b)

The claim that f is onto is true.

Proof: A function f from $\mathbb{Z}$ to $\mathbb{Z}$ is onto if for every $y \in \mathbb{Z}$, there exists $x \in \mathbb{Z}$ such that $f(x) = y$. We must show that this holds for our function f.

Let $y \in \mathbb{Z}$. Notice that $7y \in \mathbb{Z}$ as well. Now observe that

$$f(7y) = \left\lfloor \frac{7y+1}{7} \right\rfloor = \left\lfloor \frac{7y}{7} + \frac{1}{7} \right\rfloor = \left\lfloor y + \frac{1}{7} \right\rfloor = y.$$

thus for every $y \in \mathbb{Z}$ there is an $x \in \mathbb{Z}$, namely $x = 7y$, such that $f(x) = y$.

therefore f is onto □

2.2 2(b) 10 / 10

- ✓ **- 0 pts** Correct
- **- 0 pts** See comments.
- **- 1 pts** See comments
- **- 2 pts** See comments
- **- 3 pts** See comments
- **- 4 pts** See comments
- **- 5 pts** See comments

3P3=6 2P2=2
3C2=3 3C1=3

Question 3 a)

Break into steps
1: choose two fur colors to select two kittens from. can't select calico because only one kitten, so only $C(3,2)$ choices.
2: for the color with one kitten must break into cases because calico only has one kitten and other color choice has 4 kittens to choose from
case 1: calico kitten, no choices
case 2: remaining color with 4 kittens, 4 choices
3: choose kittens for two kitten colors. Each color has 4 kittens to choose from, so $C(4,2)$ choices

with calico: $C(3,2) \cdot C(4,2) \cdot C(4,2) \cdot C(1,1)$
without calico: $C(3,2) \cdot C(4,2) \cdot C(4,2) \cdot C(4,1)$

therefore in total: $C(3,2) \cdot C(4,2) \cdot C(4,2) + 4 \cdot C(3,2) \cdot C(4,2) \cdot C(4,2)$
which is equal to 540 total ways to choose 5 kittens

b) Break into cases
1. 1 cat in middle ($C(3,1)$) of the two orange. Then choose how to place 2 other cats out of the 3 choices (two on either side or one on each side) $C(3,2)$. Then choose how to order remaining cats $P(2,2)$. Finally how to order orange cats $P(2,2)$
total: $C(3,1) \cdot C(3,2) \cdot P(2,2) \cdot P(2,2)$
2. 2 cats in middle ($C(3,2)$ ways to choose them then $P(2,2)$ to order them). Then choose which side of orange cats the other cat goes $C(2,1)$. Finally order orange cats $P(2,2)$
total: $C(3,2) \cdot P(2,2) \cdot C(2,1) \cdot P(2,2)$
3. All 3 cats between orange cats. $P(3,3)$ ways. Then order orange cats $P(2,2)$ ways
total: $P(3,3) \cdot P(2,2)$

therefore in total we have $C(3,1) \cdot C(3,2) \cdot P(2,2) \cdot P(2,2) + C(3,2) \cdot P(2,2) \cdot C(2,1) \cdot P(2,2) + P(3,3) \cdot P(2,2)$
which is 72 ways to line up in total

3.1 3(a) **10 / 10**

✓ **- 0 pts** **Correct**

- 0.5 pts Correct combinatorics; arithmetic error

- 1.5 pts Picked the pairs of cats in an order, and didn't correct for overcounting

- 1.5 pts Used C(4,2) to pick a single kitten

- 1.5 pts Within each pair, picked the kittens in order

- 10 pts Completely incorrect / nothing written

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with calico: $C(3,2)\cdot C(4,2)\cdot C(4,2)\cdot C(1,1)$

without calico: $C(3,2)\cdot C(4,2)\cdot C(4,2)\cdot C(4,1)$

therefore in total: $C(3,2)\cdot C(4,2)\cdot C(4,2) + 4\cdot C(3,2)\cdot C(4,2)\cdot C(4,2)$

which is equal to 540 total ways to choose 5 kittens

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   total: $C(3,1)\cdot C(3,2)\cdot P(2,2)\cdot P(2,2)$

2. 2 cats in middle (C(3,2) ways to choose them then P(2,2) to order them). Then choose which side of orange cats the other cat goes C(2,1). Finally order orange cats P(2,2)

   total: $C(3,2)\cdot P(2,2)\cdot C(2,1)\cdot P(2,2)$

3. All 3 cats between orange cats. P(3,3) ways. Then order orange cats P(2,2) ways

   total: $P(3,3)\cdot P(2,2)$

therefore in total we have $C(3,1)\cdot C(3,2)\cdot P(2,2)\cdot P(2,2) + C(3,2)\cdot P(2,2)\cdot C(2,1)\cdot P(2,2) + P(3,3)\cdot P(2,2)$

which is 72 ways to line up in total

3.2 3(b) **10 / 10**

✓ **- 0 pts** **Correct justification of the correct answer.**

- 1 pts Correct setup, but incorrect answer due to arithmetic error.

- 2 pts Incorrect setup, but only failed to distinguish the two orange kittens.

- 4 pts Incorrect approach or justification, but shows understanding of basic counting principles.

- 6 pts Only (correctly) justified the $$5!$$ possible naive arrangements and/or the $$3!$$ possible arrangements of the non-orange kittens.

- 8 pts Had some correct explanation, but did not show understanding of basic counting principles.

- 10 pts Failed to correctly justify any important part of the problem.

Question 4

If R is antisymmetric, then $R \cap S$ is antisymmetric.

Proof: Since R is antisymmetric, if (x,y) and $(y,x) \in R$, then $x = y$.
We must show that if $(x,y),(y,x) \in R \cap S$, it means $x = y$.

Observe if $(x,y),(y,x) \in R \cap S$, this implies that $(x,y),(y,x) \in R$. Since R is antisymmetric, $x = y$.
Therefore for all $(x,y),(y,x) \in R \cap S$, $x = y$.
Thus $R \cap S$ is antisymmetric as this is the definition of antisymmetric □

Prove "if R is symmetric, then $R \cap S$ is symmetric" is false.

Proof: Show for some symmetric R that $R \cap S$ is not symmetric.
Let $R = \{(1,2),(2,1)\}$. R is symmetric. Let $S = \{(1,2)\}$.

Observe that $R \cap S = \{(1,2)\}$, which is not symmetric.
Therefore if R is symmetric, it does not mean $R \cap S$ is symmetric □

4 Problem 4 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 2 pts** Neglected to say that $x\neq y$
- **- 4 pts** Counterexample is incorrect or missing
- **- 2 pts** In the counterexample, R is not symmetric
- **- 1 pts** In the counterexample, $R \cap S$ is written incorrectly
- **- 6 pts** The proof that $R \cap S$ is antisymmetric is missing
- **- 5 pts** The proof of the first part is incorrect but the definition of antisymmetric relations is stated
- **- 1 pts** Small inconsistency in explainaitions
- **- 2 pts** A specific counter example is not provided, but it is explained how to obtain it
- **- 1 pts** In the proof of the first part, we start with $(x,y)\in R$ instead of $(x,y) \in R\cap S$
- **- 5 pts** The proof of the first part relies on R being symmetric
- **- 1 pts** The conclusion $x=y$ for $(x,y),(y,x) \in R\cap S$ is missing
- **- 10 pts** Incorrect

Question 5

$R = \{(1,1)(2,2)(3,3)(4,4)(5,5)(1,3)(3,1)(2,3)(3,2)(1,2)(2,1)\}$

Partition of X

$[1] = \{1,2,3\}$ $[2] = \{1,2,3\}$ $[3] = \{1,2,3\}$ $[4] = \{4\}$ $[5] = \{5\}$

Thus S is a partition of X given by $S = \{\{1,2,3\},\{4\},\{5\}\}$

number of equivalence classes

there are 3 equivalence classes, namely $[1]=[2]=[3]=\{1,2,3\}$, $[4]=\{4\}$ and $[5]=\{5\}$, for the relation R that I provided.

5 Problem 5 6 / 6

✓ **- 0 pts** Correct

- 2 pts Example is not reflexive

- 2 pts Example is not transitive

- 2 pts Example is not symmetric

- 1 pts Minor error(s) or step missing (see comments)

- 1 pts Did not state how many equivalence classes there are

- 2 pts Equivalence relation is not containing (1,3) or (2,3)

- 1 pts Did not say what the partition is, but did write the equivalence classes correctly somewhere in the work

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- 2 pts Only gave the equivalence classes, but did not give the equivalence relation

Question 6 Prove $C(2n, n) < 2^{2n}$ for $n \geq 1$.

Proof

Basis Step ($n=1$):

observe that $C(2,1) = \frac{2!}{1!(2-1)!} = 2! = (2)(1) = 2$

and that $2^{2(1)} = 2^2 = 4$

thus $C(2n, n) < 2^{2n}$ for $n = 1$

Inductive Step

Assume for some $n \geq 1$ that $C(2n, n) < 2^{2n}$

must show that $C(2(n+1), n+1) < 2^{2(n+1)}$

LHS:

$$C(2(n+1), n+1) = \frac{(2n+2)!}{(n+1)!(n+1)!}$$

$$= \frac{(2n+2)(2n+1)(2n)!}{(n+1)^2 n! n!}$$

$$= \frac{2(n+1)(2n+1)}{(n+1)^2} C(2n, n)$$

$$< \frac{2(2n+1)}{(n+1)} (2^{2n})$$ → by inductive assumption.

$$= \frac{(2n+1)}{(n+1)} (2^{2n+1})$$ → note that $\frac{2n+1}{n+1} < 2$ for all $n \geq 1$. See proof of this on next page (helper proof).

$$< 2(2^{2n+1})$$

$$= 2^{2(n+1)}$$

Therefore $C(2(n+1), n+1) < 2^{2(n+1)}$ for all $n \geq 1$

Thus by the principle of mathematic induction, $C(2n, n) < 2^{2n}$ for all $n \geq 1$ □

Helper proof: prove $\frac{2n+1}{n+1} < 2$ for all $n \geq 1$

basis step ($n=1$) see that $\frac{2(1)+1}{1+1} = \frac{3}{2} < 2$, Thus $\frac{2n+1}{n+1} < 2$ for $n=1$

Inductive step

Assume $\frac{2n+1}{n+1} < 2$ for some $n \geq 1$

Show $\frac{2(n+1)+1}{(n+1)+1} < 2$. This can be rewritten as $2n+3 < 2(n+2)$ expanding we get

$2n+3 < 2n+4$ which is clearly true if $n \geq 1$.

thus by the principle of mathematic induction, $\frac{2n+1}{n+1} < 2$ for all $n \geq 1$.

6 Problem 6 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 1 pts** Missing minor details/small mistakes
- **- 2 pts** Minor issues with solution
- **- 3 pts** Issues with solution
- **- 5 pts** Major issues with solution
- **- 7 pts** Some good ideas, but not a solution
- **- 8 pts** Some understanding of key concepts, but not a solution