Math 61 Homework Exam 1

REDACTED

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Problem 1

I, Raine Soriano, have read the directions and have not used any non-permitted materials nor received any help on this exam

Problem 2

Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(b) = \left\lfloor \frac{b+1}{7} \right\rfloor$.

(a) Is f one-to-one? Justify your answer with a proof.

Proof. Let $b_1 = 1$ and $b_2 = 2$. Then $f(b_1) = f(1) = \left\lfloor \frac{1+1}{7} \right\rfloor = 0$ and $f(b_2) = f(2) = \left\lfloor \frac{2+1}{7} \right\rfloor = 0$. So $f(b_1)$ is equal to $f(b_2)$, but $b_1 \neq b_2$. Thus, f is not one-to-one. □

(b) Is f onto? Justify your answer with a proof.

Proof. We must show that for all $y \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$ such that $f(b) = y$. Substituting the function for f and disregarding the floor operation (since y is an integer), we have

$$\frac{b+1}{7} = y$$
$$\implies b + 1 = 7y$$
$$\implies b = 7y - 1,$$

where $b \in \mathbb{Z}$. Thus, every $y \in \mathbb{Z}$ maps to a $b \in \mathbb{Z}$, and f is onto.

□

Problem 3

A pet shelter has a kitten room with 13 distinct kittens. Of the 13, four of them are orange, four of them are gray, four of them are black, and one of them is calico.

(a) You decide you want to take home 5 kittens, and you want two kittens with one fur color, two kittens with another fur color, and a single kitten of a third color (e.g. two distinct orange, two distinct gray, and one black). How many ways can you choose the 5 kittens?

Solution. We can split this into two cases, either we get the calico cat or not get the calico cat (since there is only 1, if we get the calico cat, it would be the single kitten of a third color). If we get the calico, there are

$$C(3,2) \times C(4,2) \times C(4,2) = 108$$

combinations. If we don't get the calico cat, there are

$$C(3,2) \times C(4,2) \times C(4,2) \times C(4,1) = 432$$

combinations. We sum these together to get

$$108 + 432 = 540$$

combinations of kittens. ☐

(b) You end up deciding to get 2 orange kittens, 2 black kittens, and 1 calico kitten. You get home and decide to take a picture of the kittens sitting in a row, but the orange kittens keep fighting each other. How many ways can you line up the kittens so that the orange kittens are not next to each other? (Individual kittens are still considered distinct.)

Solution. First we choose the 2 positions for the orange cats. Then we permute the 2 orange cats. Then we permute the remaining 3 cats. There are

$$C(5-1,2) \times P(2,2) \times P(3,3) = 6 \times 2 \times 6 = 72$$

ways to line up the kittens. ☐

Problem 4

Let R and S be relations on a set X. Prove that if R is antisymmetric, then $R \cap S$ is antisymmetric. Then provide a counterexample to show that "if R is symmetric, then $R \cap S$ is symmetric" is a false statement.

Proof. (statement 1) Suppose $(x, y) \in R \cap S$. Then $(x, y) \in R$. Since R is antisymmetric, (y, x) is not in R. Thus, (y, x) is also not in $R \cap S$, and $R \cap S$ is antisymmetric.

(statement 2) Let $R = \{(1, 2), (2, 1)\}$ and $S = \{(1, 2)\}$. Then $R \cap S = \{(1, 2)\}$. Clearly, R is symmetric, but $R \cap S$ is not symmetric. Thus the statement is false. ☐

Problem 5

Let $X = \{1, 2, 3, 4, 5\}$. Give an example of an equivalence relation R on X such that $(1, 3) \in R$, $(2, 3) \in R$, and R has at least two distinct equivalence classes. Then write down the partition given by the equivalence classes of your example. How many equivalence classes are there?

Solution. $[1] = \{1, 2, 3\}$ and $[4] = \{4, 5\}$. The partition of X is $X_p = \{\{1, 2, 3\}, \{4, 5\}\}$. There are 2 equivalence classes. ☐

Problem 6

Use the Principle of Mathematical Induction to prove that $C(2n, n) < 2^{2n}$ for all $n \geq 1$.

Proof. Let P be the statement $C(2n, n) < 2^{2n}$.

Basis Step: We set $n = 1$. We must show that P holds for $n = 1$. Observe $C(2(1), 1) = 2$ and $2^{2(1)} = 4$. Since $2 < 4$, we have verified P is true for $n = 1$.

Inductive Step: We assume that P is true for some n. We must show that P is also true for $n+1$. Observe

$$\begin{aligned}
C(2(n+1), n+1) &= C(2n+2, n+1) \\
&= \frac{(2n+2)!}{(n+1)!(n+1)!} \\
&= \frac{(2n)!(2n+1)(2n+2)}{n!n!(n+1)(n+1)} \\
&= \frac{(2n)!}{n!n!} \times \frac{(2n+1)}{n+1} \times 2 \\
&= C(2n, n) \times \frac{(2n+1)}{n+1} \times 2 \\
&< C(2n, n) \times 2 \times 2 && \text{since } \frac{a}{b} > \frac{a+1}{b+1} \\
&< 2^{2n} \times 2 \times 2 && \text{by inductive hypothesis} \\
&= 2^{2n+2} = 2^{2(n+1)}.
\end{aligned}$$

Thus, P holds for $n+1$. This concludes the inductive step.

Through the Principle of Mathematical Induction, we conclude that P is true for all $n \geq 1$.

□