22W-MATH-61-LEC-1 / 22W-MATH-61-LEC-2 Exam 1--Written Portion

TOTAL POINTS

66 / 66

QUESTION 1

1 Problem 1 0 / 0

✓ **- 0 pts Correct**

QUESTION 2

Problem 2 20 pts

2.1 2(a) 10 / 10

✓ **- 0 pts Correct**

- 5 pts Miscalculated floor function in counterexample.

- 5 pts Used non-integer inputs for $$f$$.

- 8 pts Attempted to prove that $$f$$ is one-to-one by performing invalid operations on the floor function.

- 10 pts Blank

2.2 2(b) 10 / 10

✓ **- 0 pts Correct**

- 0 pts See comments.

- 1 pts See comments

- 2 pts See comments

- 3 pts See comments

- 4 pts See comments

- 5 pts See comments

QUESTION 3

Problem 3 20 pts

3.1 3(a) 10 / 10

✓ **- 0 pts Correct**

- 0.5 pts Correct combinatorics; arithmetic error

- 1.5 pts Picked the pairs of cats in an order, and didn't correct for overcounting

- 1.5 pts Used C(4,2) to pick a single kitten

- 1.5 pts Within each pair, picked the kittens in order

- 10 pts Completely incorrect / nothing written

3.2 3(b) 10 / 10

✓ **- 0 pts Correct justification of the correct answer.**

- 1 pts Correct setup, but incorrect answer due to arithmetic error.

- 2 pts Incorrect setup, but only failed to distinguish the two orange kittens.

- 4 pts Incorrect approach or justification, but shows understanding of basic counting principles.

- 6 pts Only (correctly) justified the $$5!$$ possible naive arrangements and/or the $$3!$$ possible arrangements of the non-orange kittens.

- 8 pts Had some correct explanation, but did not show understanding of basic counting principles.

- 10 pts Failed to correctly justify any important part of the problem.

QUESTION 4

4 Problem 4 10 / 10

✓ **- 0 pts Correct**

- 2 pts Neglected to say that $$x\neq y$$

- 4 pts Counterexample is incorrect or missing

- 2 pts In the counterexample, $$R$$ is not symmetric

- 1 pts In the counterexample, $$R \cap S$$ is written incorrectly

- 6 pts The proof that $$R \cap S$$ is antisymmetric is missing

- 5 pts The proof of the first part is incorrect but the definition of antisymmetric relations is stated

- 1 pts Small inconsistency in explainaitions

- 2 pts A specific counter example is not provided, but it is explained how to obtain it

- **1 pts** In the proof of the first part, we start with $$(x,y)\in R$$ instead of $$(x,y) \in R\cap S$$
- **5 pts** The proof of the first part relies on $$R$$ being symmetric
- **1 pts** The conclusion $$x=y$$ for $$(x,y),(y,x) \in R\cap S$$ is missing
- **10 pts** Incorrect

QUESTION 5

5 Problem 5 **6 / 6**

✓ **- 0 pts Correct**

- **2 pts** Example is not reflexive
- **2 pts** Example is not transitive
- **2 pts** Example is not symmetric
- **1 pts** Minor error(s) or step missing (see comments)
- **1 pts** Did not state how many equivalence classes there are
- **2 pts** Equivalence relation is not containing (1,3) or (2,3)
- **1 pts** Did not say what the partition is, but did write the equivalence classes correctly somewhere in the work
- **2 pts** Incorrect or missing partition, and the correct equivalence classes are not written anywhere
- **0.5 pts** Notational error (see comments)
- **2 pts** Only gave the equivalence classes, but did not give the equivalence relation

QUESTION 6

6 Problem 6 **10 / 10**

✓ **- 0 pts Correct**

- **1 pts** Missing minor details/small mistakes
- **2 pts** Minor issues with solution
- **3 pts** Issues with solution
- **5 pts** Major issues with solution
- **7 pts** Some good ideas, but not a solution
- **8 pts** Some understanding of key concepts, but not a solution

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(b) = \left\lfloor \frac{b+1}{7} \right\rfloor$. *(Recall for a real number x, $\lfloor x \rfloor = x$ if x is an integer, while $\lfloor x \rfloor$ is the largest integer less than x if x is not an integer, e.g. $\lfloor 1.345 \rfloor = 1$ while $\lfloor -17 \rfloor = -17$.)*

   (a) [10 pts] Is f one-to-one? Justify your answer with a proof.

f is not one-to-one.

Proof: Consider the counterexample: $b=0$ and $b=1$, $0,1 \in \mathbb{Z}$.

$f(0) = \lfloor \frac{0+1}{7} \rfloor = 0$

$f(1) = \lfloor \frac{1+1}{7} \rfloor = 0$, so $f(0) = f(1)$.

Yet, $0 \neq 1$, so f is not one-to-one. $\square$

   (b) [10 pts] Is f onto? Justify your answer with a proof.

f is onto.

Proof: If for all $z \in \mathbb{Z}$, we can find a $b \in \mathbb{Z}$ such that $f(b) = z$, then f is onto.

So, we define a function g that provides some b such that $f(b) = z$ for each z:

$g(z) := 7z - 1$

Letting $b = 7z-1$, $f(7z-1) = \lfloor \frac{7z-1+1}{7} \rfloor = \lfloor \frac{7z}{7} \rfloor = \lfloor z \rfloor = z$.

So, for any z, we can find a b such that $f(b) = z$.

Thus, f is onto. $\square$

You do not need to write full sentences for 3(a) and 3(b). Still make sure to show your work, and make it clear what you are doing. Put your final answer in the box in completely simplified form. You can use a calculator or online calculator to get help you similplify.

3. A pet shelter has a kitten room with 13 distinct kittens. Of the 13, four of them are orange, four of them are gray, four of them are black, and one of them is calico.

   (a) [10 pts] You decide you want to take home 5 kittens, and you want two kittens with one fur color, two kittens with another fur color, and a single kitten of a third color (e.g. two distinct orange, two distinct gray, and one black). How many ways can you choose the 5 kittens?

4 O
4 G
4 B
1 C

3 fur colors belong to at least 2 kittens.
Choose 2 of these to be the colors with 2 kittens.
There are 2 remaining colors for the last kitten — either calico or the remaining color with 4 kittens.
There are $\binom{4}{2}$ ways to choose 2 kittens from 4.
There are $\binom{4}{1}$ ways to choose 1 from 4.
There is $\binom{1}{1}$ way to choose 1 from 1.

$$\binom{3}{2}\left(\binom{4}{1}+\binom{1}{1}\right)\binom{4}{2}^2 = 3(5)(6)^2 = 540$$

Put your final answer in this box: $\boxed{540}$

   (b) [10 pts] You end up deciding to get 2 orange kittens, 2 black kittens, and 1 calico kitten. You get home and decide to take a picture of the kittens sitting in a row, but the orange kittens keep fighting each other. How many ways can you line up the kittens so that the orange kittens are not next to each other? (Individual kittens are still considered distinct.)

O|O

Let orange kittens separate the others into 3 categories: to the left, between, or to the right.

Initially distribute 1 kitten between. There are 2 kittens left,
There are $\binom{2+2}{2} = \binom{4}{2} = 6$ ways to distribute them.
Order the orange kittens: $P(2,2) = 2$
Order the other 3 : $P(3,3) = 6$

$$(6)(2)(6) = 72$$

Put your final answer in this box: $\boxed{72}$

4. [10 pts] Let R and S be relations on a set X. Prove that if R is antisymmetric, then $R \cap S$ is antisymmetric. Then provide a counterexample to show that "if R is symmetric, then $R \cap S$ is symmetric" is a false statement.

R antisymmetric $\Rightarrow$ $R \cap S$ antisymmetric

Proof by contradiction:

Suppose that R is antisymmetric, yet $R \cap S$ is not antisymmetric. Then, for some $a, b \in X$, $(a,b) \in R \cap S$ and $(b,a) \in R \cap S$, where $a \neq b$.

Because $(a,b) \in R \cap S$ and $(b,a) \in R \cap S$, $(a,b) \in R$ and $(b,a) \in R$,

$a \neq b$, so we have arrived at a contradiction to our assumption that R is antisymmetric.

Thus, $R \cap S$ must be antisymmetric. $\square$

"R symmetric $\Rightarrow$ $R \cap S$ symmetric" is a false statement.

Proof: Consider the counterexample:

Let $X = \{1, 2\}$

$R = \{(1,2), (2,1)\}$

$S = \{(1,2)\}$.

$R \cap S = \{(1,2)\}$. $(1,2) \in R \cap S$ but $(2,1) \notin R \cap S$,

so $R \cap S$ is not symmetric.

Thus, R symmetric does not imply $R \cap S$ symmetric. $\square$

5. [6 pts] Let $X = \{1, 2, 3, 4, 5\}$. Give an example of an equivalence relation R on X such that $(1,3) \in R$, $(2,3) \in R$, and R has at least two distinct equivalence classes. Then write down the partition of X given by the equivalence classes of your example. How many equivalence classes are there?

R is reflexive so $(1,1), (2,2), (3,3), (4,4), (5,5) \in R$.

R is symmetric so $(3,1), (3,2) \in R$

R is transitive so $(1,2), (2,1) \in R$

$$\boxed{R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (5,5)\}}$$

Let P denote the partition of X.

$$\boxed{P = \{\{1,2,3\}, \{4\}, \{5\}\}}$$

There are 3 equivalence classes in this example.

6. [10 pts] Use the Principle of Mathematical Induction to prove that $C(2n,n) < 2^{2n}$ for all $n \geq 1$.

Proof:

We start with the base case $n=1$:

$C(2(1),1) = 2 \qquad 2<4$

$2^{2(1)} = 4 \qquad C(2(1),1) < 2^{2(1)}$

So, the statement $C(2n,n) < 2^{2n}$ is true for $n=1$.

Next, we take the inductive step.

We assume that for some $n \geq 1$, $C(2n,n) < 2^{2n}$ is true.

We want to show that $C(2(n+1),n+1) < 2^{2(n+1)}$.

Expand $C(2(n+1),n+1)$ and $C(2n,n)$

$$= \frac{(2(n+1))!}{(n+1)!\,(n+1)!} \qquad = \frac{(2n)!}{n!\,n!}$$

$$= \frac{(2n)!\,(2n+1)(2n+2)}{n!\,n!\,(n+1)(n+1)}$$

$$= C(2n,n)\left(\frac{(2n+1)(2n+2)}{(n+1)(n+1)}\right) \text{ after substitution}$$

$$= C(2n,n)\left(\frac{(2n+1)\,2}{n+1}\right)$$

$$= 2C(2n,n)\left(\frac{2n+1}{n+1}\right) < 2\cdot 2^{2n}\left(\frac{2n+1}{n+1}\right) \text{ by our assumption.}$$

$$< 2^{2n+1}\left(\frac{2n+2}{n+1}\right)$$

$$= 2^{2n+1}\cdot 2$$

$$= 2^{2n+2} = 2^{2(n+1)}$$

So, if $C(2n,n) < 2^{2n}$ is true, then $C(2(n+1),n+1) < 2^{2(n+1)}$ is true.

By induction, $C(2n,n) < 2^{2n}$ is true for all $n \geq 1$. $\square$