M61 Midterm 1
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1. I, Benson Har, have read the directions and have not used any non-permitted materials nor received any help on this exam.

2a) Proof

f is not one-to-one. Consider $b_1 = 2$ and $b_2 = 3$.

Then $f(b_1) = \left\lfloor \frac{2+1}{7} \right\rfloor = \left\lfloor \frac{3}{7} \right\rfloor = 0$

and $f(b_2) = \left\lfloor \frac{3+1}{7} \right\rfloor = \left\lfloor \frac{4}{7} \right\rfloor = 0.$

So $f(b_1) = f(b_2)$ but $b_1 \neq b_2$. Thus f is not one-to-one. $\square$

b) We must show that for every $y \in \mathbb{Z}$, there exists $x \in \mathbb{Z}$ such that $f(x) = y$. Assume $f(x) = y$. Then substituting for $f(x)$,

$$f(x) = \left\lfloor \frac{x+1}{7} \right\rfloor = y.$$

Let $r \in [0,1)$ such that $\frac{x+1}{7} - r \in \mathbb{Z}$. Then

$$\left\lfloor \frac{x+1}{7} \right\rfloor = \frac{x+1}{7} - r = y$$

$$\Rightarrow x = 7(y+r) - 1$$

Notice $7(y+r)-1$ is defined because $y \in \mathbb{Z}$. Then $x \in \mathbb{Z}$.
Thus f is onto. $\square$

3a) 13 distinct kittens.
4 orange, 4 gray, 4 black, 1 calico
What you want:
5 kittens $\to$ 2 with one color, 2 with another color, and one with a 3rd color

by color: 3 choices with at least 2 kittens
by out of color group: 4 choices for 2 kittens 2 times
by out of rest: $(13-4-4)$ choices for 1 kitten
by multiplication principle: $\binom{3}{2}\cdot\binom{4}{2}\cdot\binom{4}{2}\cdot\binom{5}{1} = 540$

There are $\boxed{540 \text{ ways}}$ to choose the 5 kittens.

3b)

x x x x x

o o
o o
o o
o o
o o
o o

6, each kitten is distinct so $6 \cdot 2 = 12$

12 ways to place 2 orange kittens
3 spots left for the rest of them
Each kitten is distinct
So $12 \cdot 3! = 72$

By Multiplication principle
$12 \cdot 3 \cdot 2 = 72$

$\boxed{72 \text{ ways}}$ you can line the kittens up this way

4. Claim: Let R and S be relations on a set X. If R is antisymmetric then $R \cap S$ is antisymmetric.
Proof: By definition of an intersection, if $x, y \in R \cap S$, then $x, y \in R$ and $x, y \in S$. Because R is antisymmetric, $x = y$. So $x = y$ for $R \cap S$. Thus, $R \cap S$ is antisymmetric if R is antisymmetric.
Claim: "if R is symmetric, then $R \cap S$ is symmetric" is false
Proof: Consider $R = \{(1,2),(2,1)\}$ and $S = \{(2,1)\}$. R is symmetric since for every $(x,y) \in R$ there exists $(y,x) \in R$.

4. cont.) Observe $R \cap S = \{(2,1)\}$.

Since $(2,1) \in R \cap S$ but $(1,2) \notin R \cap S$, $R \cap S$ is not symmetric. Thus the claim "if R is symmetric, then $R \cap S$ is symmetric" is false. $\square$

5. $X = \{1,2,3,4,5\}$

Equivalence relation $R = \left\{ \begin{array}{l} (1,3),(2,3),(3,2),(3,1),(1,1), \\ (1,2),(2,1),(2,2),(3,3),(4,5),(5,4), \\ (4,4),(5,5) \end{array} \right\}$

Partition $P = \{\{1,2,3\},\{4,5\}\}$

There are 2 equivalence classes.

6. __Claim__: $C(2n,n) < 2^{2n}$ for all $n \geq 1$

__Proof__: __Basis step__: $(n=1)$

Observe $C(2,1) = 2$ and $2^2 = 4$.

Thus $2 < 4$.

__Inductive Step__:

Assume for some $n \geq 1$ that we know $C(2n,n) < 2^{2n}$.

We must show $C(2(n+1),n+1) = 2^{2(n+1)}$

Let's start with left hand side:

$$C(2(n+1),n+1) = \frac{(2(n+1))!}{(n+1)!(2(n+1)-n+1)!} = \frac{(2(n+1))!}{(n+1)!(n+1)!} = \frac{(2n+2)!}{(n+1)!(n+1)!}$$

$$= \frac{(2n+2)(2n+1)(2n!)}{(n+1)n!(n+1)n!} = \frac{2\cancel{(n+1)}(2n+1)(2n!)}{\cancel{(n+1)}n!(n+1)n!} = \frac{2(2n+1)(2n!)}{(n!)^2(n+1)}$$

Observe $C(2n,n) = \frac{2n!}{n!(2n-n)!} = \frac{2n!}{(n!)^2}$.

Then $\frac{2(2n+1)(2n!)}{(n!)^2(n+1)} = \frac{2(2n+1)}{n+1} \cdot C(2n,n)$.

Observe $\lim_{n \to \infty} \frac{2n+1}{n+1} = 2$. Then assuming that n is some large quantity: $2 \cdot \frac{(2n+1)}{n+1} \cdot C(2n,n) = 2 \cdot 2\, C(2n,n) = 4C(2n,n)$.

Let's work with the right side:

$$2^{2(n+1)} = 2^{2n+2} = 4 \cdot 2^{2n}$$

Since $C(2n,n) < 2^{2n}$, then $4 \cdot C(2n,n) < 4 \cdot 2^n$.

This completes the induction step. Thus, $C(2n,n) < 2^n$ for $n \geq 1$ is true by the Principle of Mathematical Induction. $\square$