22W-MATH-61-LEC-1 / 22W-MATH-61-LEC-2 Final Exam--Written Portion

KEATON HEISTERMAN

TOTAL POINTS

55 / 55

QUESTION 1

1 Honesty Statement **0 / 0**

✓ **- 0 pts** Correct

QUESTION 2

2 Problem 1 **8 / 8**

✓ **- 0 pts** Correct

- 0.5 pts Small notational error

- 1 pts Small errors. See comments.

- 2 pts Proof does not cover the case when the output is 0

- 2 pts Error(s) or detail(s) missing but proof is mostly correct. See comments.

- 3 pts Outline of proof is correct, but multiple errors and/or details are missing.

- 4 pts Incorrect proof and/or missing the key details, but what is written shows some understanding of subsets and of onto

- 6 pts Proved all outputs are contained in $$\mathbb{Z}^{\geq 0}$$ instead of proving the function is onto (these are different statements)

- 6 pts Misunderstood the function, but did correctly write the definition of onto somewhere (this is not the absolute value function)

- 6 pts Incorrect proof with correct definition of onto, but argument does not show understanding of subsets/cardinalities

- 6 pts Claimed the function is onto because $$\mathcal{S}$$ has "more elements" than $$\mathbb{Z}$$ (which is not a valid argument), and did not write the correct definition of onto anywhere

- 7 pts Incorrect without understanding of onto or the function demonstrated

1 This is missing "$$=y$$".

QUESTION 3

Problem 2 10 pts

3.1 Problem 2(a) **5 / 5**

✓ **- 0 pts** Correct

- 1 pts Small error

- 2.5 pts Only gave one correct function (need a function for the vertex set and a function for the edge set)

- 2 pts Vertex function is correct, but edge function is not compatible with given vertex function

- 4 pts Both functions incorrect (or only one incorrect function given)

- 5 pts Missing or no functions written

- 2 pts Wrote sets with ordered elements. Remember sets don't have an order. This is not the correct way to define a function.

- 0.5 pts Small notational error

- 3 pts Major errors, but the correct idea is somewhat conveyed in what is written.

3.2 Problem 2(b) **5 / 5**

✓ **- 0 pts** Correct

- 1 pts Did a series reduction incorrectly, in a significant way.

- 2 pts Correctly said the graph contained something homeomorphic to $$K_5$$ or $$K_{3,3}$$, but did not show steps

- 4 pts Incorrectly said the graph is planar, but the explanation showed some understanding of concepts from the course

- 5 pts Missing or said the graph is planar with no explanation

- **- 3 pts** Does not find appropriate subgraph.
- **- 4 pts** Said graph is non-planar with no explanation.

QUESTION 4

Problem 3 12 pts

4.1 Problem 3(a) **9 / 9**

✓ **- 0 pts Correct**

Reflexive

- **- 1 pts** Small error in reflexive step
- **- 2 pts** Only showed (x,x)R(x,x), but needed to show (x,y)R(x,y)
- **- 2 pts** Incorrect argument, but what is written shows some understanding of what needs to be shown to prove a relation is reflexive.
- **- 3 pts** Missing or incorrect without understanding demonstrated

Symmetric

- **- 1 pts** Minor error in symmetric argument
- **- 2 pts** Incorrect argument, but what is written shows some understanding of what needs to be shown to prove a relation is symmetric.
- **- 3 pts** Missing or incorrect without understanding demonstrated

Transitive

- **- 1 pts** Minor error in transitive argument
- **- 2 pts** Incorrect argument, but what is written shows some understanding of what needs to be shown to prove a relation is transitive.
- **- 3 pts** Missing or incorrect without understanding of transitivity demonstrated

4.2 Problem 3(b) **3 / 3**

✓ **- 0 pts Correct**

- **- 1 pts** Missing (2,1)
- **- 1 pts** Missing one or two elements
- **- 2 pts** Missing 2-4 elements
- **- 3 pts** Missing more than 4 elements
- **- 2 pts** Includes elements not in the equivalence class

QUESTION 5

Problem 4 15 pts

5.1 Problem 4(a) **5 / 5**

✓ **- 0 pts Correct**

- **- 1 pts** Small error in counterexample
- **- 2 pts** Said the statement was false and gave a correct explanation, but did not give a counterexample
- **- 4 pts** Said the statement was false with no explanation or counterexample shown
- **- 5 pts** Missing or incorrect
- **- 1 pts** Not explained why two graphs are not isomorphic.
- **- 1 pts** An explicit counterexample is not provided. However, it is explained how to construct it.
- **- 4 pts** The counterexample is incorrect

5.2 Problem 4(b) **5 / 5**

✓ **- 0 pts Correct**

- **- 1 pts** Small error in explanation
- **- 2 pts** Argued in reverse (started with the conclusion $$P(n,r) \geq C(n, r)$$ and ended with $$r! \geq 1$$).
- **- 4 pts** Said the statement is true but did not give any explanation.
- **- 4 pts** Incorrectly said the statement was false, but what is written shows some understanding of concepts
- **- 5 pts** Missing or incorrect without understanding demonstrated

5.3 Problem 4(c) **5 / 5**

✓ **- 0 pts Correct**

- **- 0.5 pts** Justification for homeomorphic but not non-isomorphic.
- **- 0.5 pts** Justification for non-isomorphic but not homeomorphic
- **- 1 pts** Lack of justification
- **- 1 pts** Minor issues with solution

- **- 2 pts** Issues with solution
- **- 3 pts** Good understanding of relevant concepts, but not a full solution.
- **- 4 pts** Some understanding of certain concepts, but not a solution.
- **- 5 pts** Incorrect

QUESTION 6

6 Problem 5 **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Didn't use strong induction
- **- 3 pts** Choice of n in basis step is too small or too large
- **- 2 pts** Used that c_n is increasing in n without proof
- **- 1 pts** Needed to cover more cases in basis step (else inductive hypothesis cannot be applied)
- **- 1 pts** Small problem with statement of inductive hypothesis
- **- 2 pts** Big problem with statement of inductive hypothesis
- **- 5 pts** Big problems with proof-writing
- **- 2 pts** Backwards derivation

Keaton Heisterman 905-537-242

1B: Kedar Karhadkar, Thurs 9am

I, Keaton Heisterman, have ~~and~~ read the directions and have not used any non-permitted materials nor recieved any help on this exam

1. we must show that forall $y \in Z^{\geq 0}$, there exists a $X \in S$ such that $f(x) = |X|$ to prove that f is onto.

Proof: $(y=0), (y=1)$
Observe that the empty set is a subset of $\mathbb{Z}$. If $X_0 = \{\}$, $X_0 \in S$ and $f(X_0) = |X_0| = 0$. Thus this equality holds for the basis step $y=0$

~~Inductive step: Assume for some $n \geq 1$, $n \in \mathbb{Z}^{\geq 0}$ that $y = n$ and that there exists some~~

$(y=1)$
Define X_y to be $\{1, 2, ..., y\}$. Thus $X_1 = \{1\}$. Observe that

$f(X_1) = |X_1| = 1 = y$. Thus there exists $X_1 \in S$ such that $f(X_1) = 1$.

Inductive Step: Assume $y \geq 1$. Define X_y to be $\{1, 2, ..., y\}$. Assume that there exists $X_y \in S$ such that $f(X_y) = y$. Show that there exists $X_{y+1} \in S$ such that $f(X_{y+1}) = y+1$.

Notice that $X_{y+1} = X_y \cup \{y+1\}$ and that $\{y+1\}$ is not a subset of X_y since X_y does not have element $y+1$. because of this,

$|X_y \cup \{y+1\}| = |X_y| + |\{y+1\}| = y+1$ through our inductive assumption.

Therefore by the principle of mathematic induction we have proved that for all $y \in \mathbb{Z}^{\geq 0}$, There exists $X_y \in S$ such that $f(X_y) = y$ □

By this proof we see that for all $y \in \mathbb{Z}^{\geq 0}$, there is a solution such that $f(X) = y$. Therefore f is onto by definition.

1 Honesty Statement **0 / 0**

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2 Problem 1 **8 / 8**

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- 0.5 pts Small notational error

- 1 pts Small errors. See comments.

- 2 pts Proof does not cover the case when the output is 0

- 2 pts Error(s) or detail(s) missing but proof is mostly correct. See comments.

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- 6 pts Claimed the function is onto because $$\mathcal{S}$$ has "more elements" than $$\mathbb{Z}$$ (which is not a valid argument), and did not write the correct definition of onto anywhere

- 7 pts Incorrect without understanding of onto or the function demonstrated

1 This is missing "$$=y$$".

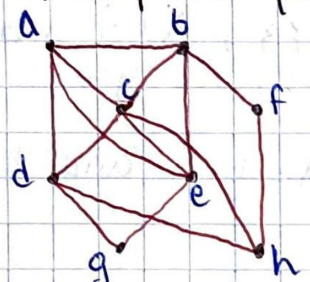
2. a) f is bijection from vertex set of G_1 to G_2. g is bijection from edge set of G_1 to G_2.

$f: G_1 \to G_2$ defined by: $a \to x$, $b \to y$, $c \to z$, $d \to w$, $e \to v$.

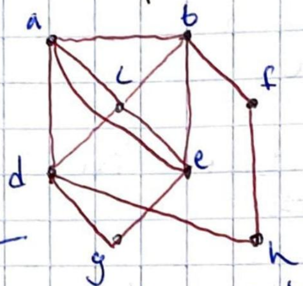
$g: G_1 \to G_2$ defined by: $e_1 \to h_2$, $e_2 \to h_6$, $e_3 \to h_7$, $e_4 \to h_1$, $e_5 \to h_5$

$e_6 \to h_3$, $e_7 \to h_4$

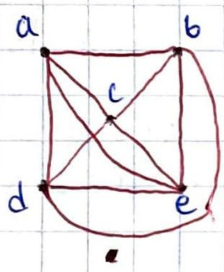
b) Grap is not planar. Find subgraph homeomorphic to K_5



remove edges (c,h) →



from here we perform series reduction to connect d to e and d to b. remove g and connect d to e and remove h and f to connect d to b.



This is the graph K_5... this is the subgraph homeomorphic to K_5.

3.1 Problem 2(a) **5 / 5**

✓ **- 0 pts** **Correct**

- 1 pts Small error

- 2.5 pts Only gave one correct function (need a function for the vertex set and a function for the edge set)

- 2 pts Vertex function is correct, but edge function is not compatible with given vertex function

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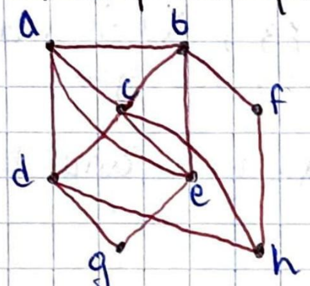
b) Grap is not planar. Find subgraph homeomorphic to K_5



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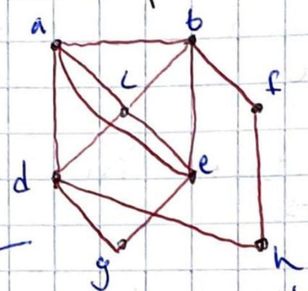

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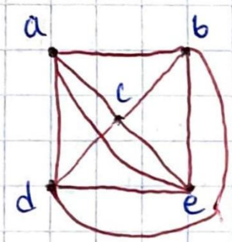


This is the graph K_5... this is the subgraph homeomorphic to K_5.

3.2 Problem 2(b) **5 / 5**

✓ **- 0 pts** **Correct**

- 1 pts Did a series reduction incorrectly, in a significant way.

- 2 pts Correctly said the graph contained something homeomorphic to $$K_5$$ or $$K_{3,3}$$, but did not show steps

- 4 pts Incorrectly said the graph is planar, but the explanation showed some understanding of concepts from the course

- 5 pts Missing or said the graph is planar with no explanation

- 3 pts Does not find appropriate subgraph.

- 4 pts Said graph is non-planar with no explanation.

3. a) Proof: R must be transitive, symmetric and reflexive to be an equivalence relation

Transitive * for all of these, selected pairs are any pair in $X \times X$ *

for any $(a,b),(c,d),(e,f) \in X \times X$

Assume $(a,b)R(c,d)$ and $(c,d)R(e,f)$. Show $(a,b)R(e,f)$:
Observe that since $(a,b)R(c,d)$, $a-c=b-d$ and that since $(c,d)R(e,f)$, $c-e=d-f$,

$c=d+e-f$, $a-c=b-d$

$$a-d-e+f=b-d \Rightarrow a-e=b-f$$

by the definition of R, since $a-e=b-f$, $(a,b)R(e,f)$.

Thus R is transitive

Symmetric for any $(a,b),(c,d) \in X \times X$

Assume $(a,b)R(c,d)$. Show $(c,d)R(a,b)$. We know $a-c=b-d$.
Observe that for $(c,d)R(a,b)$ to be true, $c-a$ must equal $d-b$.
We can re-write this to be:

$$c-a=d-b$$
$$-c+a \quad -c+a$$
$$0=d-b+a-c$$
$$-d+b \quad -d+b$$
$$b-d=a-c$$

→ Since $(a,b)R(c,d)$, we know this is true.
Thus $(c,d)R(a,b)$.

This means R is symmetric.

Reflexive for any $(a,b) \in X \times X$

We must show that $(a,b)R(a,b)$.
Notice that if $(a,b)R(a,b)$, $a-a=b-b$.
$0=0$ → This is true.
Therefore $(a,b)R(a,b)$. Thus R is reflexive.

Since R is transitive, symmetric and reflexive, R is an equivalence relation □.

4.1 Problem 3(a) **9 / 9**

✓ **- 0 pts** **Correct**

Reflexive

- 1 pts Small error in reflexive step

- 2 pts Only showed (x,x)R(x,x), but needed to show (x,y)R(x,y)

- 2 pts Incorrect argument, but what is written shows some understanding of what needs to be shown to prove a relation is reflexive.

- 3 pts Missing or incorrect without understanding demonstrated

Symmetric

- 1 pts Minor error in symmetric argument

- 2 pts Incorrect argument, but what is written shows some understanding of what needs to be shown to prove a relation is symmetric.

- 3 pts Missing or incorrect without understanding demonstrated

Transitive

- 1 pts Minor error in transitive argument

- 2 pts Incorrect argument, but what is written shows some understanding of what needs to be shown to prove a relation is transitive.

- 3 pts Missing or incorrect without understanding of transitivity demonstrated

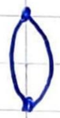
3.

b) $[(2,1)] = \{(2,1), (3,2), (4,3), (5,4), (6,5), (7,6)\}$

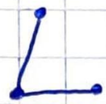
4.

a) False. Let P be the invariant "has 2 edges".

G_1



G_2



Both G_1 and G_2 satisfy the property P, but G_1 and G_2 are not isomorphic because G_2 has invariant "has 3 vertices" and G_1 does not.

b) True. Notice that $C(n,r) = \frac{P(n,r)}{r!}$ for positive integers n, r with $n \geq r$

$$P(n,r) \geq \frac{P(n,r)}{r!}$$

$$r! \cdot P(n,r) \geq P(n,r).$$

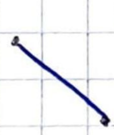
since $r!$ is always greater than or equal to 1 we have:

$r! \cdot P(n,r) \geq P(n,r)$ for all r, n such that $n \geq r$ and n, r are positive integers.

c) False. Graphs G_1 and G_2 are homeomorphic (perform series reduction on G_1) but they are not isomorphic since G_1 has 3 vertices and G_2 does not, and "has 3 vertices" is an invariant.

G_1

G_2



4.2 Problem 3(b) **3 / 3**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Missing (2,1)
- **- 1 pts** Missing one or two elements
- **- 2 pts** Missing 2-4 elements
- **- 3 pts** Missing more than 4 elements
- **- 2 pts** Includes elements not in the equivalence class

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b) $[(2,1)] = \{(2,1),(3,2),(4,3),(5,4),(6,5),(7,6)\}$

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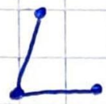
G_1  G_2 

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G_1  G_2 

5.1 Problem 4(a) 5 / 5

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- 1 pts Small error in counterexample

- 2 pts Said the statement was false and gave a correct explanation, but did not give a counterexample

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- 1 pts Not explained why two graphs are not isomorphic.

- 1 pts An explicit counterexample is not provided. However, it is explained how to construct it.

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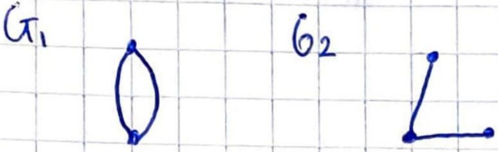

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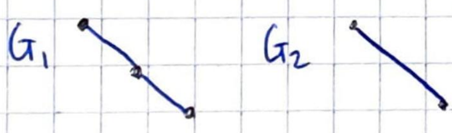

5.2 Problem 4(b) **5 / 5**

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- **- 1 pts** Small error in explanation
- **- 2 pts** Argued in reverse (started with the conclusion $$P(n,r) \geq C(n, r)$$ and ended with $$r! \geq 1$$).
- **- 4 pts** Said the statement is true but did not give any explanation.
- **- 4 pts** Incorrectly said the statement was false, but what is written shows some understanding of concepts
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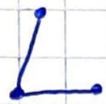
G_1

G_2



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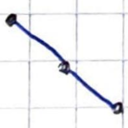

G_2

5.3 Problem 4(c) **5 / 5**

- ✓ **- 0 pts** **Correct**
- **- 0.5 pts** Justification for homeomorphic but not non-isomorphic.
- **- 0.5 pts** Justification for non-isomorphic but not homeomorphic
- **- 1 pts** Lack of justification
- **- 1 pts** Minor issues with solution
- **- 2 pts** Issues with solution
- **- 3 pts** Good understanding of relevant concepts, but not a full solution.
- **- 4 pts** Some understanding of certain concepts, but not a solution.
- **- 5 pts** Incorrect

5. Strong Mathematical Induction Proof

basis step ($n=3$)

Note that $C_1 = 1 \cdot C_0 = 1 \cdot 1 = 1$

Observe: $C_3 = 3 \cdot C_1 = 3$ and $3! = 6$. Thus $C_3 < 3!$

This concludes the basis step.

Inductive Step Assume $n \geq 3$. Also assume for ~~some~~ all $3 \leq k \leq n$ that $C_k < k!$ Show that $C_{n+1} < (n+1)!$

$(n+1)! > C_{n+1}$

$= (n+1) \cdot C_{\lfloor \frac{n+1}{2} \rfloor}$

Break into 2 cases:

$\lfloor \frac{n+1}{2} \rfloor < 3$ ($n=3, n=4$) → $n=3$ case covered in basis step.

Note $C_2 = 2 \cdot C_1 = 2$

($n=4$):

$C_4 = 4 \cdot C_2 = 8$ and $4! = 24$ thus $4! > C_4$

$\lfloor \frac{n+1}{2} \rfloor \geq 3$ → observe $\lfloor \frac{n+1}{2} \rfloor < n$ for $n \geq 3$

$C_{n+1} = (n+1) \cdot C_k$

$C_{n+1} < (n+1) \cdot k!$ → Since $3 \leq k \leq n$, $(n+1) \cdot k! \leq (n+1) \cdot n! = (n+1)!$

$C_{n+1} < (n+1)!$

This concludes inductive step.

Thus by principle of Mathematic Induction we have proved

$C_n < n!$ for $n \geq 3$ □

6 Problem 5 **10 / 10**

✓ **- 0 pts** **Correct**

- 2 pts Didn't use strong induction

- 3 pts Choice of n in basis step is too small or too large

- 2 pts Used that c_n is increasing in n without proof

- 1 pts Needed to cover more cases in basis step (else inductive hypothesis cannot be applied)

- 1 pts Small problem with statement of inductive hypothesis

- 2 pts Big problem with statement of inductive hypothesis

- 5 pts Big problems with proof-writing

- 2 pts Backwards derivation