MATH 61 Final

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Student Information

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Discussion Section: 1A

Honesty Statement

I, Jason Cheng, have read the directions and have not used any non-permitted materials nor received any help on this exam.

Problem 1

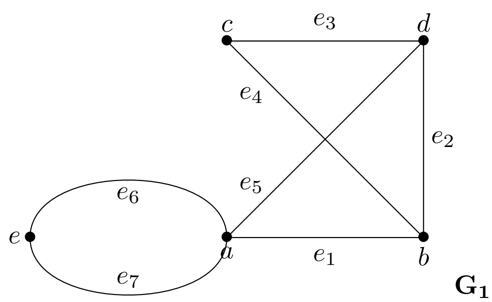
Let $\mathcal{S}$ denote the set of all finite subsets of $\mathbb{Z}$ and let $\mathbb{Z}^{\geq 0}$ denote the set of all nonnegative integers. Define a function $f : \mathcal{S} \to \mathbb{Z}^{\geq 0}$ by $f(X) = |X|$. Prove that f is onto.

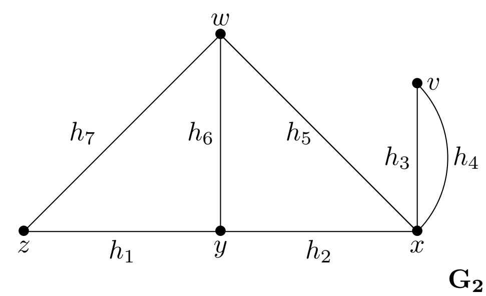
Proof. For every $y \in \mathbb{Z}^{\geq 0}$, $X = [1, y] \cap \mathbb{Z}$ is a subset of $\mathbb{Z}$ that satisfies $f(X) = |X| = y$. Therefore, f is onto. $\square$

Problem 2

Use the given graphs to answer the questions.

(a) The graphs G_1 and G_2 shown below are isomorphic. Write down an isomorphism from G_1 to G_2. You do not have to justify your work in this problem; just write down an isomorphism.



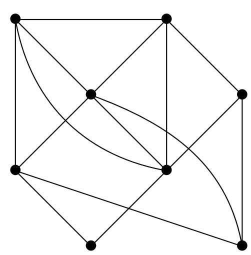


Solution.

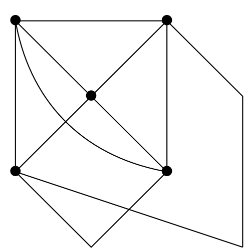
$$f = \{(e,v),(a,x),(c,z),(b,y),(d,w)\}$$
$$g = \{(e_6,h_3),(e_7,h_4),(e_5,h_5),(e_1,h_2),(e_2,h_6),(e_3,h_7),(e_4,h_1)\}$$

□

(b) Determine whether or not the following graph is planar. If the graph is planar, redraw it so that no edges cross; otherwise, find a subgraph homeomorphic to either K_5 or $K_{3,3}$.



Solution. The following graph is obtained from the original graph by three series reductions and eliminating two edges.



Since every pair of vertices are connected by an edge, this graph is K_5. Since the original graph contains a subgraph homeomorphic to K_5, by Kuratowski's Theorem, it is not planar. □

Problem 3

(a) Let $X = \{1, 2, 3, 4, 5, 6, 7\}$. Define a relation R on $X \times X$ by $(a, b)R(c, d)$ if $a - c = b - d$. Prove that R is an equivalence relation.

Proof. Let $(a, b), (c, d), (e, f)$ be any three ordered pairs in $X \times X$.

$a - a = b - b = 0$, so $(a, b)R(a, b)$. Therefore, R is reflexive.

If $(a, b)R(c, d)$ and $(c, d)R(e, f)$, then $a - c = b - d$ and $c - e = d - f$. Adding the two equations results in $a - e = b - f$, which implies that $(a, b)R(e, f)$. Since $(a, b)R(c, d) \wedge (c, d)R(e, f) \implies (a, b)R(e, f)$, R is transitive.

If $(a, b)R(c, d)$, then $a - c = b - d$. Multiplying both sides by -1 results in $c - a = d - b$, which implies that $(c, d)R(a, b)$. Since $(a, b)R(c, d) \implies (c, d)R(a, b)$, R is symmetric.

Since R is reflexive, transitive, and symmetric, R is an equivalence relation. □

(b) List all elements in $[(2, 1)]$. You do not have to justify your answer for this part; just list the elements.

Solution.

$$[(2, 1)] = \{(2, 1), (3, 2), (4, 3), (5, 4), (6, 5), (7, 6)\}$$

□

Problem 4

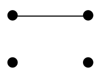
For parts (a)–(d), determine if the following statements are true or false. If the statement is true, explain why. If the statement is false, give a counterexample.

(a) Let P be an invariant. If G_1 and G_2 are two graphs that both satisfy P, then G_1 and G_2 are isomorphic.

Solution. The statement is false. Counterexample: The number of vertices in a graph is an invariant. The following two graphs satisfy the invariant, but they are not isomorphic.

G_1

G_2



□

(b) For all positive integers n, r with $n \geq r$, we have that $P(n,r) \geq C(n,r)$.

Solution. The statement is true. Reasoning:

$$\begin{aligned} P(n,r) - C(n,r) &= \frac{n!}{(n-r)!} - \frac{n!}{r!(n-r)!} \qquad \text{by the definitions of permutation and combination} \\ &= \frac{n!}{(n-r)!}\left(1 - \frac{1}{r!}\right) \end{aligned}$$

$$\begin{gathered} n > 0 \implies n! > 0 \\ n \geq r \implies n - r \geq 0 \implies (n-r)! > 0 \\ r > 0 \implies r! \geq 1 \implies 1 - \frac{1}{r!} \geq 0 \end{gathered}$$

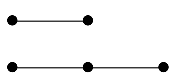
Thus, $P(n,r) - C(n,r) \geq 0 \implies P(n,r) \geq C(n,r)$. □

(c) Let G_1 and G_2 be two graphs. If G_1 and G_2 are homeomorphic, then G_1 and G_2 are isomorphic.

Solution. The statement is false. Counterexample: The following two graphs are homeomorphic because the middle vertex in G_2 can be reduced to make it isomorphic to G_1, but they are not isomorphic because they violate the invariant of having the same number of vertices.

G_1

G_2



□

Problem 5

Consider the sequence $c_0, c_1, c_2, \ldots$ defined by $c_0 = 1$, and for $n \geq 1$,

$$c_n = n \cdot c_{\lfloor n/2 \rfloor}.$$

Use mathematical induction to prove that $c_n < n!$ for $n \geq 3$.

Proof. Base cases:

$$\begin{gathered} c_3 = 3 \cdot c_1 = 3 \cdot 1 \cdot c_0 = 3 < 3! = 6 \\ c_4 = 4 \cdot c_2 = 4 \cdot 2 \cdot c_1 = 8 < 4! = 24 \\ c_5 = 5 \cdot c_2 = 10 < 5! = 120 \end{gathered}$$

Inductive hypothesis: For all $n > 3$, assuming that $c_k < k!$ for all $3 \leq k < n$, we will prove that $c_n < n!$.

If $n = 4$ or $n = 5$, we have already manually proved it in the base cases. Next, we will prove the hypothesis for $n > 5$.

$$\begin{aligned} c_n - n! &= n \cdot c_{\lfloor n/2 \rfloor} - n! && \text{by the definition of } c_n \\ &< n \cdot \lfloor n/2 \rfloor! - n! && n > 5 \implies \lfloor n/2 \rfloor \geq 3, \text{ so by the inductive hypothesis, } c_{\lfloor n/2 \rfloor} < \lfloor n/2 \rfloor! \\ &= n \left(\lfloor n/2 \rfloor! - (n-1)! \right) && n > 5 \implies \lfloor n/2 \rfloor < n + 1 \implies \lfloor n/2 \rfloor! - (n-1)! < 0 \\ &< 0 \end{aligned}$$

By the Strong Form of Mathematical Induction, $c_n < n!$ for all $n \geq 3$. $\square$