Student name: Gene Anyeung
Student ID: 503481206

MATH 61 (Butler)
Midterm II, 12 November 2008

This test is closed book and closed notes, with the exception that you are allowed one $8\frac{1}{2}'' \times 11''$ page of handwritten notes. No calculator is allowed for this test. For full credit show all of your work (legibly!), unless otherwise specified. Each problem is worth 10 points.

1. There is a huge bin filled with many balls of many different colors. What is the smallest number n of balls you need to take from the bin so that among the n balls you have taken you *must* have either seven balls of one color or seven balls with no two the same color.

(1)

all (2) are differnt

(Your answer needs two parts, first show that with n balls you must satisfy the condition, second show that with $n-1$ balls you might not satisfy the condition.)

| | | | | | | |

7 balls + 1 will not satisfy condition (2). ?

~~7+6~~.

+2

$\boxed{7 \text{ will}}$

7 balls ~~with~~ might satisfy condition (1)

7 - 1 balls with not

$\Rightarrow \boxed{n = 7}$

1	2
2	7
3	10
4	5
5	10
Σ	34

-3

2. Consider a variation of the tower of Hanoi problem where we have three poles A, B and C and start with a stack of n different sized discs on pole A arranged going from the bottom to the top by largest to smallest. We again consider the problem of moving the discs from pole A to pole C. There are three rules:

(i) We can only move one disc at a time.

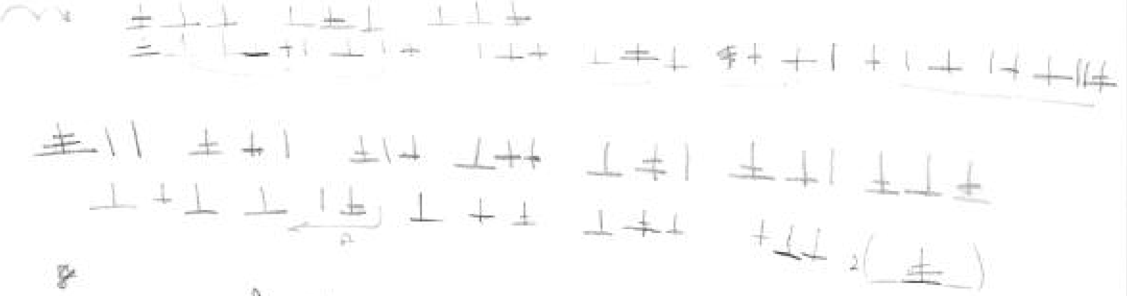
(ii) We can never put a larger disc over a smaller disc.

(iii) We can not move a disc directly from A to C or from C to A.

So for example, if $n = 1$ we now need two moves to transfer from pole A to pole C, first move from A to B and second move from B to C.

(a) Let R_n be the minimal number of moves needed to move all n discs from pole A to pole C. Set up a recurrence for R_n, explain your reasoning behind the recurrence.

n	R_n
1	2
2	8
3	26



$8 + 1 + 8 + 1 + 8 = 26$

i) have to move $n-1$ to end → R_{n-1}

ii) move biggest (n) by one → 1

iii) move $n-1$ back to first tower → R_{n-1}

iv) move biggest (n) by one → 1

v) move $n-1$ back to last tower → R_{n-1}

$\Rightarrow 3R_{n-1} + 2 \qquad R_1 = 2$

(b) Solve the recurrence in part (a).

$R_n = 3R_{n-1} + 2$

$= 3(3R_{n-2} + 2) + 2 = 3^2 R_{n-2} + 3 \cdot 2 + 2$

$= 3(3(3R_{n-3} + 2) + 2) + 2) = 3^3 R_{n-3} + 3 \cdot 3 \cdot 2 + 3 \cdot 2 + 2$

$= 3^k R_{n-k} + 2(3^{k-1} + 3^{k-2} + \cdots + 3^{k-k})$

let $k = n-1$

$R_n = 3^{n-1} R_1 + 2(3^{n-2} + 3^{n-3} + \cdots + 1)$

$= 2(3^{n-1} + 3^{n-2} + 3^{n-3} + \cdots + 1)$

← good idea, but need to finish

K_n has $\binom{n}{2}$ edges

3. Use the complete graph K_{m+n} to give a combinatorial proof that

$$\binom{m+n}{2} = \binom{m}{2} + mn + \binom{n}{2}.$$

Make sure to explain what you are counting on both sides.

---

a complete graph K_n has $\binom{n}{2}$ edges. (each edge involves a 2-element-subset of the set of all vertices)

$\Rightarrow$ K_{m+n} has $\binom{m+n}{2}$ edges.

---

if we split K_{m+n} into 2 by taking m vertices as one set V_m and remaining n as another set. V_n

consider edges only within V_m itself. there are $\binom{m}{2}$ edges

consider edges only within V_n itself. there are $\binom{n}{2}$ edges

now consider edges only between V_m and V_n, as if V_m and V_n form a bipartite graph.

there are mn edges in $K_{m,n}$

$\Rightarrow$ there are mn edges between V_m and V_n.

---

adding up, we get $\binom{m}{2} + mn + \binom{n}{2}$ edges

4. Solve the recurrence relation

$$a_n = 2\sqrt{(a_{n-1} + a_{n-2})(a_{n-1} - a_{n-2})} \quad \text{for } n \geq 2,$$

with initial conditions $a_0 = 1$, $a_1 = 2$.

-5

$a_3 = 2\sqrt{3 \cdot 1} = 2\sqrt{3}$

$a_4 = 2\sqrt{(2\sqrt{3}+2)(2\sqrt{3}-2)}$

$a_4 = 2\sqrt{8}$

$a_5 = 2\sqrt{32 - 12}$

$= 2\sqrt{20}$

$a_6 = 2\sqrt{80 - 32}$

$= 2\sqrt{48}$

$\frac{a_n^2}{4} = (a_{n-1} + a_{n-2})(a_{n-1} - a_{n-2})$

$= a_{n-1}^2 - a_{n-2}^2$

$a_n^2 = 4(a_{n-1}^2 - a_{n-2}^2)$

$a_n = 2\sqrt{a_{n-1}^2 - a_{n-2}^2}$

~~not~~ linear

not homogeneous

so can't do this →

$\frac{r^n}{r^{n-2}} = 2\sqrt{\frac{r^{n-1}r^{n-1} - r^{n-2}r^{n-2}}{r^{n-2}}}$

$r^2 = 2\sqrt{\frac{r^{2n-2}}{r^{2n-4}} - \frac{r^{2n-4}}{r^{2n-4}}} = 2\sqrt{r^2 - 1}$

$2n-2 - 2n+4$

$\Leftrightarrow \frac{r^4}{4} = r^2 - 1 \Leftrightarrow$

$r^4 = 4r^2 - 4$

$r^4 - 4r^2 + 4 = 0$

$r^2(r^2 - 4) + 4 = 0$

inner relaton

(inside sq-root)

n	a_n
1	3
2	8
3	20
4	48

$a_n = a_{n-1} + 2^n$

let $b_n = b_{n-1}$

$r = 1$

$b_n = A$

$a_n = b_n + C2^n = b_{n-1} + C(2^{n-1}) + 2^n$

$\Rightarrow 0 = -C2^n + C2^{n-1} + 2^n$

$= C2^{n-1}(-2+1) + 2^n$

$= 2^n - C2^{n-1}$

$C2^{n-1} = 2^n$

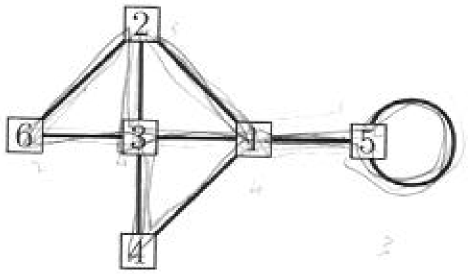
$C = \frac{2^n}{2^{n-1}} = 2$

$\Rightarrow a_n = A + 2$

$a_1 = 3 = A + 2 \Rightarrow A = 1$

$\Rightarrow a_n =$

5. A graph has an Eulerian cycle if there is a closed walk which uses each edge *exactly* once. A related problem is to find the shortest closed walk (i.e., using the fewest number of edges) which uses each edge *at least* once. (This is known as the "Chinese Postman" problem and comes up frequently in applications for optimal routing.) Consider the following graph on six vertices.



(a) Find a walk that starts and ends at vertex 1 of length 11 (i.e., using 11 edges) so that each edge in the graph is used at least once.

(1, 5, 5, 1, 4, 3, 6, 2, 3, 1, 2, 1)

(b) Explain why 11 is the smallest number of edges needed to find a walk that starts and ends at vertex 1 and uses each edge at least once.

there are 9 edges in the graph

vertices 1, 3, 4, 6 have degrees that are even.

vertices 2 and 5 have odd degrees. (but ignore 5 b/c is not simple)

so we cannot just goto and leave vertex 2 $\frac{d(2)}{2}$ times

we have to do so $\left\lceil \frac{d(2)}{2} \right\rceil$ = 2 times, going back just to use the extra edge it has. and come back, reusing an edge

b/c there is an edge between 1 and 5 is a bridge.

we must reuse edge 1-5 ~~at~~ at least once.

9 edges + 2 reused = 11.