Math 61, Lec 1
Winter 2016
Exam 2
2/22/16
Time Limit: 50 Minutes

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Discussion Section: 1F

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You may *not* use books, notes, or any calculator on this exam.

Unless otherwise stated in the problem, you may leave all answers in terms of $\binom{n}{k}$, $P(n,k)$, $k!$, or any sum, difference, product, or quotient of such symbols.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score	
1	24	16	8
2	16	16	
3	22	12	10
4	18	18	
5	20	20	
Total:	100	82	

1. (24 points) There are 999,999 natural numbers less than one million. We write any of them as a six digit number, including leading zeros. (For example, 001124 is how we write the number 1124).

   (a) How many of these numbers have all different digits?

$$P(10,6) = \frac{10!}{4!}$$

+8

   (b) How many of these numbers have exactly four distinct digits? (For example, 922433 is valid, but 922435 is not valid and 922444 is not valid).

pick 4 #s : $P(10,4)$

pick which are duplicated : $C(4,2)$

order the 6 numbers : $\frac{6!}{2!2!}$

what if only 1 duplicated

$$\text{total} = P(10,4) \cdot C(4,2) \cdot \frac{6!}{2! \cdot 2!}$$

+4

   (c) How many of these numbers have digits that sum to 18?

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$ s.t. $\cancel{x_2}$ $0 \le x_i \le 9$

≤ 9

$$C(n+k-1, k-1) = C(23, 5)$$

+4

2. (16 points) Solve the recurrence relation $A_n = 3A_{n-1} + 4A_{n-2}$, where $A_0 = 3$, $A_1 = 7$.

$A_n = 3A_{n-1} + 4A_{n-2}$

$\frac{A_n}{A_{n-2}} - \frac{3A_{n-1}}{A_{n-2}} + \frac{4A_{n-2}}{A_{n-2}} = 0$

$t^2 - 3t - 4 = 0$

$(t-4)(t+1) = 0$

$t = 4, -1$

$A_n = br_1^n + dr_2^n$

$A_n = b4^n + d(-1)^n$

$A_0 = 3 = b4^0 + d(-1)^0$

$3 = b + d$

$A_1 = 7 = b4^1 + d(-1)^1$

$7 = 4b - d$

$\left.\begin{array}{l} 3 = b + d \\ 7 = 4b - d \end{array}\right\}$

$10 = 5b$

$b = 2, \ d = 1$

$$\boxed{\therefore A_n = 2 \cdot 4^n + (-1)^n}$$

3. (22 points) Recall that a k-cycle is a cycle that includes k edges. In this problem, you will prove Ramsey's theorem, which states that if $n \geq 6$ and we color each edge of K_n either blue or red, then there must exist either a set of three blue edges that form a 3-cycle, or a set of three red edges that form a 3-cycle.
To this end, let $n \geq 6$ be arbitrary, and suppose every edge in K_n is colored either blue or red. Let v_1 be a vertex in K_n.

Prove that at least three of the edges incident to v_1 are the same color.

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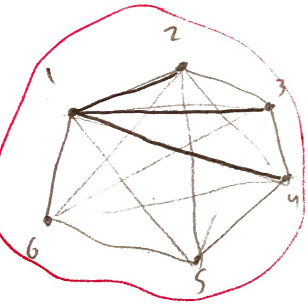
Since K_n has $n-1$ edges incident to each v_i, and $n \geq 6$, each v_i has at least 5 incident edges. ~~For~~ each edge can be colored red or blue. That leaves us w/ combinations BBBBB, RBBBB, RRBBB, RRRBB, RRRRB, and RRRRR. Each option has at least 3 of the same color. Thus, this is true for $n=6$. For $n \geq 6$, the same logic applies. Formally, if in $n-1$ edges, r are red, then $(n-1)-r$ are blue. Thus, $r + (n-1) - r \geq 6$, which means either $r \geq 3$ or $(n-1)-r \geq 3$. QED.

Combinations don't describe the situation! Order matters; since permuting vertices yields different graphs!

(b) In the previous part, you proved that at least three of the edges incident to v_1 are the same color. Without loss of generality, you may assume that color is blue. Suppose that $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_4\}$ are blue edges. Prove that between these four vertices, there must exist either a blue 3-cycle or a red 3-cycle.



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Since each vertex has at least 3 edges of the same color and each vertex is connected to every other vertex directly, we can let at least three of the remaing edges at each vertex be blue and the rest be red (or viceversa). Then, since each ~~e~~ vertex is connected to each other and there are at least three edges of the same color to each vertex, there must be a cycle back to vertex 1, 2, or 3, or 4 such that each edge is the same color.

but the color could change depending on the vertex!

This isn't enough. The fact that ~~[illegible]~~ all vertices satisfy property in (a) ~~[illegible]~~ doesn't describe ~~[illegible]~~ where a 3-cycle is.

4. (18 points) Prove the combinatorial identity

$$\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i) = C(m+n+k-1, k)$$

using a combinatorial argument. No more than half credit will be awarded to an algebraic proof. (Hint: Use Pirates and Gold.)

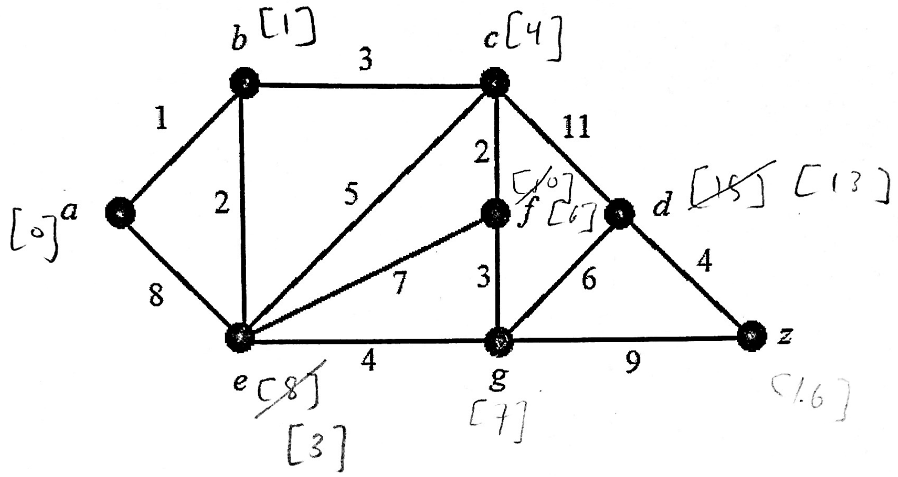
We can count the following set

1. Partion the set of m red pirates and n blue pirates that get k pieces of gold by k, the amount of gold there is. Using MP, we can pick n pirates that get i gold ($i \le k$) and m pirates that get $k-i$ gold, separately by pirates and gold. Then, using AP, we can add the disjoint partions of each $i \le k$. Thus, the set can be counted as

$$\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i)$$

2. Directly count the number of m red pirates and ~~blue~~ n blue pirates that get k pieces of gold. By pirates and gold, that quantity is $C(\underbrace{m+n}_{\text{total \# of pirates}}+k-1, k)$

5. (20 points) Run Dijkstra's algorithm on the following graph to find the shortest path from a to z. Recall that at each stage of Dijkstra's algorithm, one vertex is chosen and given a permanent label which represents the length of the shortest path from a to that vertex. Write down the list of vertices in the order in which they are given permanent labels. Additionally, find the length of a shortest path from a to z.



permanent Labels: a [0] → b[1] → e[3] → c[4] → f [6] → g [7] — ...

... → d[13] → z [16]

length from a to z: 16