Math 61, Lec 1
Winter 2016
Exam 2
2-22-16
Time Limit: 50 Minutes

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Discussion Section: 1F

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You may *not* use books, notes, or any calculator on this exam.

Unless otherwise stated in the problem, you may leave all answers in terms of $\binom{n}{k}$, $P(n,k)$, $k!$, or any sum, difference, product, or quotient of such symbols.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score	
1	16	16	
2	18	15	3
3	22	22	
4	20	20	
5	24	20	4
Total:	100	93	

1. (16 points) Solve the recurrence relation $A_n = 3A_{n-1} + 4A_{n-2}$, where $A_0 = 4$, $A_1 = 6$.

c.p. $t^2 - 3t - 4 = (t-4)(t+1)$ $r_1 = 4$ $r_2 = -1$

$A_n = b\,4^n + d\,(-1)^n$

$$\left.\begin{array}{l} 4 = b + d \\ 6 = 4b - d \\ 10 = 5b \\ b = 2 \quad d = 2 \end{array}\right\} \quad A_n = 2(4^n + (-1)^n)$$

2. (18 points) Prove the combinatorial identity

$$\sum_{i=0}^{k} C(m+k-i-1,k-i)\cdot C(n+i-1,i) = C(m+n+k-1,k)$$

using a combinatorial argument. No more than half credit will be awarded to an algebraic proof. (Hint: Use Pirates and Gold.)

~~The rh~~ They both count way to put k balls in m+n boxes.

right hand side

The rhs counts all the possible ways to put k balls in m+n boxes. think of ~~b~~ m+n boxes like $\underbrace{\quad\quad}_{m}\underbrace{\quad\quad}_{n}$. We are bound to put some $0 \le i \le k$ balls in the m label box and the rest k-i balls in the n label box [The lhs sums all ways to put i balls in

left hand side

n label and k-i balls in m label while iterating through all sizes of i up to k. This in effect counts all the ways to put k balls in m+n boxes]

~~wh~~ how do you count these?

which terms, specifically?

3. (22 points) Recall that a k-cycle is a cycle that includes k edges. In this problem, you will prove Ramsey's theorem, which states that if $n \geq 6$ and we color each edge of K_n either blue or red, then there must exist either a set of three blue edges that form a 3-cycle, or a set of three red edges that form a 3-cycle.
To this end, let $n \geq 6$ be arbitrary, and suppose every edge in K_n is colored either blue or red. Let v_1 be a vertex in K_n.

Prove that at least three of the edges incident to v_1 are the same color.

Since v_1 is a vertex in K_n, by definition, it is incident to $n-1$ edges. Argue by contradiction. If v_1 did not have three incident edges of the same color, it has at most 2 red and 2 blue edges, meaning the most edges that could be incident on v_1 is 4. However, $n \geq 6$ and $n-1 \geq 5$, meaning v_1 has at least 5 edges incident. So by the pigeonhole principle, there must be at least 3 edges of the same color incident on v_1.

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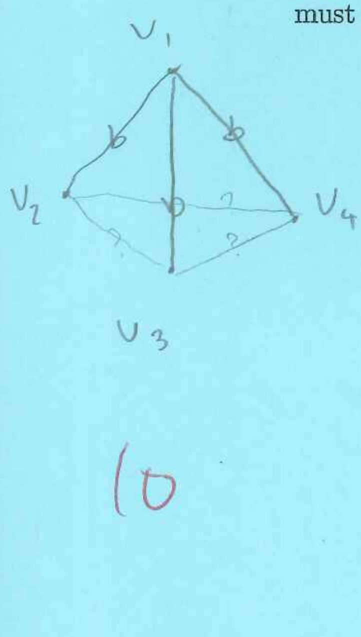
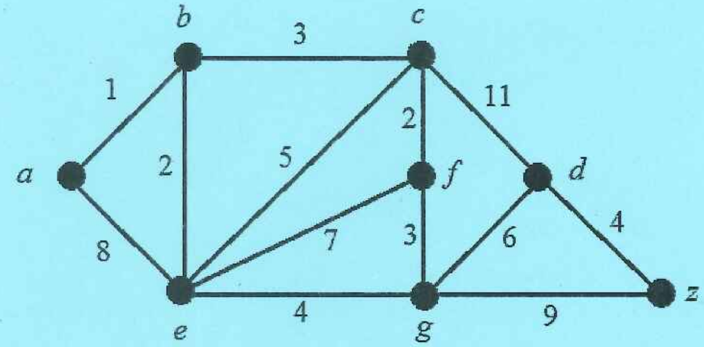
(b) In the previous part, you proved that at least three of the edges incident to v_1 are the same color. Without loss of generality, you may assume that color is blue. Suppose that $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_4\}$ are blue edges. Prove that between these four vertices, there must exist either a blue 3-cycle or a red 3-cycle.



In order to complete a blue 3-cycle, we need any of $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_2\}$ to be a blue edge. If any of the three is a blue edge, then we have a blue cycle. The only way we don't have blue cycle is if the three edges all are red edges. Then the three edges make a red 3-cycle. Therefore, there must be either a blue 3-cycle or a red 3-cycle.

10

4. (20 points) Run Dijkstra's algorithm on the following graph to find the shortest path from a to z. Recall that at each stage of Dijkstra's algorithm, one vertex is chosen and given a permanent label which represents the length of the shortest path from a to that vertex. Write down the list of vertices in the order in which they are given permanent labels. Additionally, find the length of a shortest path from a to z.



	1	2	3	4	5	6	7
a	b	e	c	f	g	d	z
	1	~~8~~ 3	4	~~10~~ 6	7	~~15~~ 13	16

a b e c f g d z

shortest path : a b e g z

length : 16

5. (24 points) There are 999,999 natural numbers less than one million. We write any of them as a six digit number, including leading zeros. (For example, 001124 is how we write the number 1124).

   (a) How many of these numbers have all different digits?

10 choices: $\cancel{9}\ \cancel{8}\ \cancel{7}\ \cancel{6}\ \cancel{5}\ \cancel{4}$ / $9\ 8\ 7\ 6\ 5$

~~9·8·7·6·5·4~~

$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$

$\frac{10!}{4!}$

+8

   (b) How many of these numbers have digits that sum to 18?

6 boxes, 18 balls $C\binom{18+6-1}{6-1} = C\binom{23}{5}$

+4

   (c) How many of these numbers have exactly four distinct digits? (For example, 922433 is valid, but 922435 is not valid and 922444 is not valid).

we need to - choose the 4 digits → $10 \cdot 9 \cdot 8 \cdot 7$

- choose the repeating digits (either 1 dig rep 3 or 2 dig rep 2

- choose the orientation of the digits

• 4 to choose which 1 dig to rep. 3 times or 4·3 to see which 2 dig to rep. 2 times

orientation $\frac{6!}{3!\cdot 1!\cdot 1!\cdot 1!} = \frac{6!}{3!}$

orientation $\frac{6!}{2!\cdot 2!\cdot 1!\cdot 1!} = \frac{6!}{2!2!}$

$\frac{10!}{6!}\left(4\frac{6!}{3!} + 4\cdot 3\frac{6!}{2!\cdot 2!}\right) = \frac{10!}{6!}\left(4 \cdot \frac{6!}{3!} + 4 \cdot 3 \cdot \frac{6!}{2!2!}\right)$

+8