Math 61, Lec 1
Winter 2016
Exam 2
2-22-16
Time Limit: 50 Minutes

Name (Print): ______________________

Name (Sign): ______________________

Discussion Section: ______________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You may *not* use books, notes, or any calculator on this exam.

Unless otherwise stated in the problem, you may leave all answers in terms of $\binom{n}{k}$, $P(n,k)$, $k!$, or any sum, difference, product, or quotient of such symbols.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score	
1	16	16	
2	18	18	
3	22	22	
4	20	20	
5	24	20	4
Total:	100	96	

1. (16 points) Solve the recurrence relation $A_n = 3A_{n-1} + 4A_{n-2}$, where $A_0 = 4$, $A_1 = 6$.

$A_n = 3A_{n-1} + 4A_{n-2}$, $A_0 = 4$ / $A_1 = 6$

$\therefore A_n - 3A_{n-1} - 4A_{n-2} = 0$

(take the characteristic polynomial)

$\therefore t^2 - 3t - 4 = 0$

$t^2 + t - 4t - 4 = 0$

$t(t+1) - 4(t+1) = 0$

$\therefore (t+1)(t-4) = 0 \quad \therefore t = \{-1, 4\}$

$\therefore A_n = c_1(-1)^n + c_2(4)^n$

$A_0 = c_1 + c_2 = 4 \quad -(1)$ $\therefore$ Add (1) and (2)

$A_1 = -c_1 + 4c_2 = 6 \quad -(2)$ $\quad 5c_2 = 10 \therefore c_2 = 2$

$\therefore c_1 = 2$

$\therefore \boxed{A_n = 2(-1)^n + 2(4)^n}$

2. (18 points) Prove the combinatorial identity

$$\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i) = C(m+n+k-1, k)$$

using a combinatorial argument. No more than half credit will be awarded to an algebraic proof. (Hint: Use Pirates and Gold.)

RHS.

Take $(m+n)$ pirates who have to split k gold in some manner
Let $x_1 \ldots x_{m+n}$ each represent a pirate's share of gold.
$\therefore x_1 + x_2 + \ldots x_n + x_{n+1} \ldots + x_{m+n} = k$

$\therefore$ We know this is $\binom{m+n+k-1}{k}$.

LHS

- We divide the pirates into the first n and the next m
$\therefore x_1 + \ldots + x_n$ and $x_1 + \ldots + x_m$

$\therefore$ Let's say for one case, the first n pirates get a total of i gold. $\therefore x_1 + \ldots + x_n = i$

Then $\binom{n+i-1}{i}$

This would mean the next m pirates get a total of $k-i$ gold $x_1 + \ldots + x_m = k-i$

Then $\binom{m+(k-i)-1}{k-i}$

y count
same
y.

$\therefore$ For that one case, $\binom{m+k-i-1}{k-i} \cdot \binom{n+i-1}{i}$ is all possibilities where the first n pirates get i gold and the next m pirates get $k-i$ gold.

We want to do this for all possible is hence we take all cases. Since the cases are discrete, i.e. the first n pirates can't have i and $i+1$ gold at the same time, we sum all the cases to get

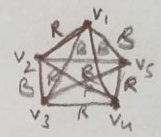
$\sum^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i)$ which is LHS

3. (22 points) Recall that a k-cycle is a cycle that includes k edges. In this problem, you will prove Ramsey's theorem, which states that if $n \geq 6$ and we color each edge of K_n either blue or red, then there must exist either a set of three blue edges that form a 3-cycle, or a set of three red edges that form a 3-cycle.

To this end, let $n \geq 6$ be arbitrary, and suppose every edge in K_n is colored either blue or red. Let v_1 be a vertex in K_n.

Prove that at least three of the edges incident to v_1 are the same color.

∴ For K_n where $n=5$



given on board,

Each v has $\delta = 4$ ∴ 4 edges leave it. Hence like the one given, For each vertex, there exist a possibility where of the 4 edges, 2 are red and 2 are blue

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For $n \geq 6$, (δ of vertices in K_n) ≥ 5

Hence each vertex will have at least 5 edges incident to it.

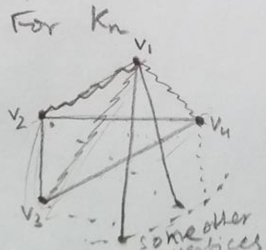
∴ By pigeon hole principle $\frac{n}{k} = 2.5$ minimum
n → no of edges adjacent to v
k → no of colors

Hence there must be a 3-cycle formed as there is at least one v with 3-edges of the same color incident to it. (which we take as v_1)

(b) In the previous part, you proved that at least three of the edges incident to v_1 are the same color. Without loss of generality, you may assume that color is blue. Suppose that $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_4\}$ are blue edges. Prove that between these four vertices, there must exist either a blue 3-cycle or a red 3-cycle.

For K_n

10



∴ Given that edges V_1V_2, V_1V_4 and V_1V_3 are blue, the other edges between these 4 vertices are V_2V_4, V_2V_3 and V_3V_4.

Case 1
If all the other 3 edges are red, we form a red 3-cycle (V_2, V_4, V_3, V_2)

Case 2a: If V_2V_4 is blue
Blue 3-cycle (V_1, V_4, V_2, V_1) formed

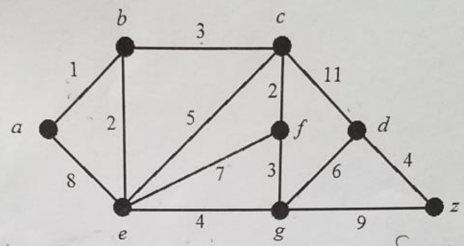
Case 2b: If V_2V_3 is blue
Blue 3-cycle (V_1, V_2, V_3, V_1) formed

Case 2c: If V_3V_4 is ...

Case 3 and up:
If more than 1 edge of V_2V_4, V_2V_3 and V_3V_4 is blue then case 2 still stands

Proved

4. (20 points) Run Dijkstra's algorithm on the following graph to find the shortest path from a to z. Recall that at each stage of Dijkstra's algorithm, one vertex is chosen and given a permanent label which represents the length of the shortest path from a to that vertex. Write down the list of vertices in the order in which they are given permanent labels. Additionally, find the length of a shortest path from a to z.



$L(a)=0;\ l(b)=1,\ l(e)=8$

$L(b)=1;\ l(e)=\min(8,3)=3,\ l(c)=4$

$L(e)=3;\ l(g)=7;\ l(f)=10;\ l(c)=8$

$L(c)=4;\ l(f)=\min(10,6)=6;\ L(d)=15$

$L(f)=6;\ l(g)=\min(7,9)=7$

$L(g)=7;\ l(d)=\min(15,13)=13;\ l(z)=16$

$L(d)=13;\ l(z)=\min(16,17)=16$

$L(z)=16.$

$\{a, b, e, c, f, g, d, z\}$

order of permane
Label

$\therefore$ Shortest path $= \{a, b, e, g, z\} = 16$ len

5. (24 points) There are 999,999 natural numbers less than one million. We write any of them as a six digit number, including leading zeros. (For example, 001124 is how we write the number 1124).

(a) How many of these numbers have all different digits?

$\underline{10} \times \underline{9} \times \underline{8} \times \underline{7} \times \underline{6} \times \underline{5}$ numbers

or ${}^{10}P_6$ because

123456] are differe[nt]
632541] numbers

+8

(b) How many of these numbers have digits that sum to 18?

$\therefore \binom{6+18-1}{6-1} - 6\binom{5+8-1}{8}$

$\therefore$ $\overline{x_1}$ $\overline{x_2}$ $\overline{x_3}$ $\overline{x_4}$ $\overline{x_5}$ $\overline{x_6}$

$\therefore x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$ +8

wher for any i, $9 \leq x_i \leq 0$

Only one x_i can ever be > 9 at a time as sum is 18. $\therefore$ let $x_i = 10$ $\therefore$ new sum $= 18 - 10 = 8$

(c) How many of these numbers have exactly four distinct digits? (For example, 922433 is valid, but 922435 is not valid and 922444 is not valid).

924333

$\therefore \dfrac{{}^{10}P_6}{2!2!} + \dfrac{{}^{10}P_6}{3!}$ +4

For all numbers with 2 sets of double numbers

For all numbers with 1 set of triple numbers.