Math 61, Lec 1
Winter 2016
Exam 1
1/25/16
Time Limit: 50 Minutes

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Discussion Section: 1F

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You may *not* use books, notes, or any calculator on this exam.

If your answer contains a number that is impossible to simplify without the use of a calculator, such as e^3, $\ln(3)$ or $\sin(3)$, you may leave answers in terms of e, $\ln$, or trig functions.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	7	7
2	18	17
3	25	25
4	25	25
5	25	12
Total:	100	86

1. (7 points) Negate the following implication:
   "If you are on the wait list, then you will be enrolled in the class."

You are on the wait list, but you won't be enrolled in the class

2. (18 points) (a) Let X be a set with n elements. How many different relations on X are there?

Think of an $n \times n$ matrix, each entry is a potential relation. Since there are n^2 entries, there are n^2 different could-be relations. Since each entry is either relatation or not a relation, there are 2^{n^2} relations.

(b) Let X be a set with n elements. How many different relations on X are there that are not reflexive?

Think of the $n \times n$ matrix again, If we know that the relations can't be reflexive, we know that the middle diagonal $\begin{bmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{bmatrix}$ is always 0, or not related

we ~~to~~ have $n^2 - n$ entries that could be related or not related (Read: two choices) so there ~~are~~ $2^{n^2} - 2^{n^2-n}$ relations on X that are not reflexive.

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(c) Let X be a set with n elements and let Y be a set with m elements, where $m \geq n$. How many different one-to-one functions from X to Y are there?

n m

function = every element n has to have one unique m

n_1 has m choices

n_2 has $m-1$ choices

n_n has $m-(n-1)$ choices

$$m \cdot (m-1) \cdot (m-2) \cdots (m-(n-1)) = \frac{m!}{(m-(n-1)-1)!}$$

$$= \frac{m!}{(m-n+1-1)!} = \boxed{\frac{m!}{(m-n)!}}$$

3. (25 points) Use mathematical induction to prove that $1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2$ for every integer $n \geq 0$.

Base case: $(2 \cdot 0 + 1) = (0+1)^2$

$1 = 1$ ✓

Induction step:

Assume for some integer $n \geq 0$, $1+3+5+\cdots+(2n+1) = (n+1)^2$

we want to prove

$$\underbrace{1+3+\cdots+(2n+1)}_{\text{we know that this}=(n+1)^2} + (2(n+1)+1) = ((n+1)+1)^2 = (n+2)^2 ✓$$

$$(n+1)^2 + (2n+3)$$

$$n^2+2n+1+2n+3$$

$$n^2+4n+4$$

$$(n+2)^2 ✓$$

since our induction hypothesis holds true for some $n \geq 0$

therefore, for every $n \geq 0$, $1+3+5+\cdots+(2n+1) = (n+1)^2$

4. (25 points) Let X be a set. We define the *power set* $\mathcal{P}(X)$ to be the set of all subsets of X. For example, if $X = \{1,2\}$, then $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

   Define a relation R on $\mathcal{P}(\mathbb{Z})$ by $(S,T) \in R$ if and only if $S \subseteq T$, for any sets S and T in $\mathcal{P}(\mathbb{Z})$. Prove that R is a partial order.

partial order : reflexive, transitive, and antisymmetric

we want to prove all 3

prove reflexive

want to show

- we w.t.s that $S \subseteq S$ for all $S \in P(\mathbb{Z})$

✓ we can see that this is true due to definition of $\subseteq$, where if the sets are identical, it is seen to be $\subseteq$

prove transitive

- we w.t.s that if $A \subseteq B$ and $B \subseteq C$, $A \subseteq C$ for any

✓ A, B, C in $P(\mathbb{Z})$. This is also true by definition, since if $A \subseteq B$, all elements in A are found in B, and if $B \subseteq C$ all elements of B, including all elements of A, are found in C so therefore all elements of A are found in C, so $A \subseteq C$

prove antisymmetric

- we w.t.s that if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

✓ If $A \subseteq B$, all elements of A are found in B, but B may contain more elements. However, if both $A \subseteq B$ and $B \subseteq A$, we know that all elements of B must be in A, so B cannot have more elements, and A and B must be equal.

Since all three is true, R is partial order

5. (25 points) As in the previous problem, for any set X, we define the *power set* $\mathcal{P}(X)$ to be the set of all subsets of X.
Consider the sets $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$.
Define a function $f : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \cup Y)$ by $f((S,T)) = S \cup T$, for $S \in \mathcal{P}(X)$ and $T \in \mathcal{P}(Y)$. Prove that f is a bijection.

bijection : 1-to-1 and onto

prove: 1-to-1

we w.t.s. that if $f((X_1,Y_1)) = f((X_2,Y_2))$ then $(X_1,Y_1)=(X_2,Y_2)$

since X and Y do not overlap, every $X \cup Y$ is unique to a combination ~~of unique X and Y sets~~. So if $f((X_1,Y_1)) = f((X_2,Y_2))$, it implies $X_1 \cup Y_1 = X_2 \cup Y_2$, and that is only possible if $X_1 = X_2$ and $Y_1 = Y_2$

Why

Therefore f is 1-to-1.

prove: onto

we w.t.s that for any set A in $X \cup Y$, there is a $S \in P(X)$ and $T \in P(Y)$ such that $f((S,T)) = A$

We know that all sets in $P(X \cup Y)$ must only contain elements found in X and Y exactly once. since the domains $P(X)$ and $P(Y)$ contains all possible subsets of X and Y, it is possible, with the union of $P(X)$ and ~~$P(Y)$~~ to have any combination of elements in $X \cup Y$. Therefore, for every set $A \in X \cup Y$, ~~there~~ is a $S \in P(X)$ and $T \in P(Y)$ such that $f((S,T)) = A$, therefore f is onto.

?

What

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Since f is both 1-to-1 and onto, it is a bijection